Supplementary Materials: Multivariate left-censored Bayesian model for predicting exposure using multiple chemical predictors

Caroline Groth[1], Sudipto Banerjee[2*], Gurumurthy Ramachandran[3], Mark R. Stenzel[4], and Patricia A. Stewart[5]

[1] Department of Preventive Medicine, Feinberg School of Medicine, Northwestern University, Chicago, Illinois 60611

[2] Department of Biostatistics, University of California-Los Angeles, Los Angeles, California 90095.

[3] Department of Environmental Health and Engineering, Bloomberg School of Public Health, Johns Hopkins University, Baltimore, Maryland 21205.

[4] Exposure Assessments Applications, LLC, Arlington, Virginia 22207

[5] Stewart Exposure Assessments, LLC, Arlington, Virginia 22207

[*] *email:* sudipto@ucla.edu

Contents

1 RJAGS Model

```
model { for (i in 1:N){

is.notcensoredx1[i]~dinterval(X1[i],cx1[i])

X1[i]~dnorm(beta[SEG[i],1], tausqx1[SEG[i]])


is.notcensoredx2[i]~dinterval(X2[i],cx2[i])

X2[i]~dnorm(mu1[i], tausqx2[SEG[i]])

mu1[i]<-beta[SEG[i],2]+beta[SEG[i],3]*X1[i]


is.notcensoredx3[i]~dinterval(X3[i],cx3[i])

X3[i]~dnorm(mu2[i], tausqx3[SEG[i]])

mu2[i]<-beta[SEG[i],4]+beta[SEG[i],5]*X1[i]+beta[SEG[i],6]*X2[i]

}


mu.beta1[1:2] ~dmnorm(mean1[],prec1[,])

mu.beta2[1:3] ~dmnorm(mean2[],prec2[,])

Omega1[1:2,1:2] ~ dwish(W1[,],v1)

Omega2[1:3,1:3] ~ dwish(W2[,],v2)


for(k in 1:NSEG) {

beta[k,1] ~ dnorm(0,0.00001)

beta[k,2:3] ~ dmnorm(mu.beta1[],Omega1[,])

beta[k,4:6] ~ dmnorm(mu.beta2[],Omega2[,])

tausqx1[k] ~ dgamma(0.01,0.01)

tausqx2[k] ~ dgamma(0.01,0.01)

tausqx3[k] ~ dgamma(0.01,0.01)
```

```
sigmax1[k]<-1/tausqx1[k]

sigmax2x1[k]<-1/tausqx2[k]

sigmax3x1x2[k]<-1/tausqx3[k]

}

LSigma1[1:2,1:2]<-inverse(Omega1[,])

LSigma2[1:3,1:3]<-inverse(Omega2[,])

}
```

2 MCMC Sampler in R

```
MultivariateModel=function(M,p,Inits,Data,n){


#Results to Output
n.betas=sum(seq(1,p,length=p))
variancematrix=matrix(,nrow=M,ncol=p)
covariance=matrix(,nrow=M,ncol=(p*p-p))
correlationresults=matrix(,nrow=M,ncol=(p*p))
variancecovarianceline=matrix(,nrow=M,ncol=(p*p))
beta.new=matrix(,nrow=M,ncol=n.betas)
new.sigma=matrix(,nrow=M,ncol=p)
Xmatrix=matrix(,nrow=M,ncol=(n*p))



#Data
#Defining Data
Xfinal=Data$Xfin
LODXs=Data$LODs
is.notcensoredidicators=Data$is.notcensored

dataset=data.frame(cbind(Xfinal,LODXs))

namestotakefrom=paste0("X",1:p)
namestotakefromlod=paste0("XLOD",1:p)
names(dataset)=c(namestotakefrom[1:p],namestotakefromlod[1:p])
mu.beta1=Data$mu.beta1
Sigma.beta1=Data$Sigma.beta1
Sigma.beta.p=Data$Sigma.beta1
mu.beta.p=Data$mu.beta1



#Inits
Xinits=Inits$Xinits
Xmatrix[1,]=c(Xinits)
beta.init=Inits$beta.init
sigma.init=Inits$sigma.init

betamatrix=matrix(beta.init,nrow=p,ncol=p)
beta.new[1,]=rep(beta.init,n.betas)
```

```
new.sigma[1,]=rep(sigma.init,p)
Variablematrix=dataset[,1:p]


################################################################################
#Multivariate Code Sampler:
#M is the total number of iterations+1
time=proc.time()
#####################################################################################################
#Overarching Gibbs Sampler
#####################################################################################################

for(m in 2:M){
betamatrix[1,1:1]=beta.new[m-1,1]

n.betafunc=function(g){
n.beta=sum(seq(1,g,length=g))
return(n.beta)
}
n.beta=sapply(1:p,n.betafunc)


#Make a betamatrix to use in calculations
for(f in 2:p){
betamatrix[f,1:f]=beta.new[m-1,(n.beta[f-1]+1):(n.beta[f])]
}

#Taus are conditional variances
tau=c(new.sigma[(m-1),])

#Define X Matrix; design matrix Xd
Xu=matrix(Xmatrix[m-1,],nrow=n,ncol=p,byrow=FALSE)
conditionalmean=matrix(,nrow=n,ncol=p)
Xd=cbind(rep(1,n),Xu)


#calculate conditional mean
condmeanfunc=function(s){
conditionalmeanres=Xd[,1:s]%*%betamatrix[s,1:s]
```

```
return(conditionalmeanres)
}
conditionalmeanp1=sapply(2:p,condmeanfunc)
conditionalmean=cbind(NA,conditionalmeanp1)
conditionalmean[,1]=betamatrix[1,1]

X1samplesfunc=function(i){
X1topm1func=function(u){
#Calculating part of mean for truncated normal
parts=function(r){
partu=betamatrix[r,u+1]*(Xd[i,r+1]-(conditionalmean[i,r]-Xd[i,u+1]*betamatrix[r,u+1]))/tau[r]
return(partu)
}

possiblepart=seq(p,2)
upp=length(possiblepart)

prt=function(o){
non=o-1
return(parts(p-non))
}

minib=sum(sapply(1:upp,prt))
bt=(conditionalmean[i,u]/tau[u])+minib
variancevec=1/tau
Bt=solve(t(variancevec[u:p])%*%c(1,betamatrix[((u+1):p),(u+1)]^2))
#if Xs are censored, sample X from truncated normal
if(is.na(dataset[i,u])==TRUE){
Xnsample=rtruncnorm(1,Bt*bt,sd=sqrt(Bt),a=-Inf,b=dataset[i,p+u])
}

#if Xs are noncensored, fill in datapoint
if(is.na(dataset[i,u])==FALSE){
Xnsample=dataset[i,u]
}
return(Xnsample)
}

Xnsampleres=sapply(1:(p-1),X1topm1func)
return(Xnsampleres)
```

```
}
Xsamplesres=sapply(1:n,X1samplesfunc)
Xd[,(2:p)]=t(Xsamplesres)

#Sample final censored Xps

Xnsamples=function(i){
#if Xs are censored; sample from truncated normal
if(is.na(dataset[i,p])==TRUE){
Xdsamp=Xd[i,-(p+1)]
Xnsample=rtruncnorm(1,t(Xdsamp)%*%betamatrix[p,],sd=sqrt(tau[p]),a=-Inf,b=dataset[i,(p*2)])
}
#if Xs not censored,fill in datapoint
if(is.na(dataset[i,p])==FALSE){
Xnsample=dataset[i,p]
}
return(Xnsample)
}

Xd[,(p+1)]=sapply(1:n,Xnsamples)



allx=function(d){
allx=Xd[,(1+d)]
return(allx)
}

allx=unlist(sapply(1:p,allx))
Xmatrix[m,]=c(allx[,1:p])

#Sample regression coefficients for X2 to Xp
#Sample conditional variances
for(h in 2:p){
Y=Xd[,h+1]
Xdesign=as.matrix(Xd[,1:h])
mu.beta=rep(mu.beta.p,h)
Sigma.beta=diag(Sigma.beta.p,h)
Vx=as.matrix(tau[h]*diag(1,n))
Bx=solve(t(Xdesign)%*%solve(Vx)%*%Xdesign+solve(Sigma.beta))
bx=solve(Sigma.beta)%*%mu.beta+t(Xdesign)%*%solve(Vx)%*%Y
```

```
beta.hat=Bx%*%bx

n.betafunc=function(g){

n.beta=sum(seq(1,g,length=g))

return(n.beta)

}

n.beta=sapply(1:p,n.betafunc)

#sample regression coefficients (multivariate normal sample)

beta.new[m,(n.beta[h-1]+1):(n.beta[h])]=as.numeric(rmvnorm(1,beta.hat,Bx))

#sample variances

sigmapart=t(Y-Xdesign%*%beta.new[m,(n.beta[h-1]+1):n.beta[h]])%*%(Y-Xdesign%*%beta.new[m,(n.beta[h-1]+1):n.beta[h]])

new.sigma[m,h]=1/rgamma(1,a+n/2,b+0.5*as.numeric(sigmapart))

}

#Sample X1 variance and mean

J=rep(1,n)

Vx1=tau[1]

B1=solve((n/Vx1)+(1/Sigma.beta1))

b1=solve(Sigma.beta1)%*%mu.beta1+solve(Vx1)*t(J)%*%Xd[,2]

beta.new[m,1]=rnorm(1,B1*b1,sqrt(B1))

sigmapart1=as.numeric(t(Xd[,2]-J*beta.new[m,1]))%*%as.numeric((Xd[,2]-J*beta.new[m,1]))

new.sigma[m,1]=1/rgamma(1,a+n/2,b+0.5*sigmapart1)


##############################################################################################################

#Calculating Correlation, Variance Covariance Matrix, and saving them

#These could be written as functions to speed up calculations, but if p is not too large,

#then this will be fast

#Variance/Covariance Calculations

variancematrix[m,1]=new.sigma[m,1]

covariance[m,1]=variancematrix[m,1]*beta.new[m,3]

variancematrix[m,2]=new.sigma[m,2]+

covariance[m,1]*solve(variancematrix[m,1])*covariance[m,1]

n.betafunc=function(g){

n.beta=sum(seq(1,g,length=g))

return(n.beta)

}
```

```
n.beta=sapply(1:(p+1),n.betafunc)
resvar1=c(variancematrix[m,1],covariance[m,1],variancematrix[m,2])

variancecovarianceline[m,]=c(variancematrix[m,1],covariance[m,1],
variancematrix[m,2],rep(NA,((p*p)-length(resvar1))))

#Develop Variance/Covariance Matrix
if(p>2){

for(w in 2:(p-1)){

n.beta=sapply(1:(p+1),n.betafunc)
diagmat1=matrix(,nrow=w,ncol=w)
diagmat1[1,]=c(variancecovarianceline[m,1],rep(0,(w-1)))
for(q in 2:w){
if(q<w){
diagmat1[q,]=c(variancecovarianceline[m,(n.beta[q-1]+1):(n.beta[q])],
rep(0,w-q))
}
if(q==w){
diagmat1[q,]=c(variancecovarianceline[m,(n.beta[q-1]+1):(n.beta[q])])
}
}

variancecovariance=as.matrix(forceSymmetric(diagmat1,signature(from = "ddiMatrix",
to = "symmetricMatrix")))
covariance[m,(n.beta[w-1]+1):(n.beta[w-1]+w)]=
as.matrix(variancecovariance)%*%beta.new[m,(n.beta[w]+2):(n.beta[w]+1+w)]
variancematrix[m,(w+1)]=new.sigma[m,w+1]+
covariance[m,(n.beta[w-1]+1):(n.beta[w-1]+w)]%*%solve(variancecovariance)%*%covariance[m,
(n.beta[w-1]+1):(n.beta[w-1]+w)]
variancecovarianceline[m,(n.beta[w]+1):(n.beta[w+1])]=
c(covariance[m,(n.beta[w-1]+1):(n.beta[w-1]+w)],variancematrix[m,(w+1)])

}
}

for(v in 2:p){
variancecovariancefinal=matrix(,nrow=v,ncol=v)
variancecovariancefinal[1,]=c(variancecovarianceline[m,1],rep(0,(v-1)))
```

```
for(q in 2:v){
if(q<v){
variancecovariancefinal[q,]=
c(variancecovarianceline[m,(n.beta[q-1]+1):(n.beta[q])],rep(0,v-q))
}
if(q==v){
variancecovariancefinal[q,]=
c(variancecovarianceline[m,(n.beta[q-1]+1):(n.beta[q])])
}
}
}

finishedvariancecovariance=forceSymmetric(variancecovariancefinal,signature(from = "ddiMatrix", to = "symmetricMatrix"))
correlationmatrixfinal=cov2cor(finishedvariancecovariance)

#Correlation results table
createcorrelationresultstable=function(l){
rescor=correlationmatrixfinal[l,]
return(rescor)
}
correlationresults[m,]=unlist(sapply(1:p,createcorrelationresultstable))




} #End of Sampler

return(list(variancecovarianceline=variancecovarianceline,variancematrix=variancematrix,covariance=covariance,
new.sigma=new.sigma,
beta.new=beta.new,n.betas=n.betas,correlationresults=correlationresults,
Xmatrix=Xmatrix))

}
```

3 WAIC$_Y$

If the primary interest is modeling the response Y, one can simplify WAIC and rely only on the likelihood component for Y, referred to as WAIC$_Y$. The calculations and formulas are the same as when we consider the full likelihood, but now L_{ij} is the likelihood for only Y_i at iteration j (observation i). This formulation of WAIC allows us to know which model is better modeling our response Y. It also more easily allows models with different numbers of chemical covariates to be compared without assuming other models for the covariates not included in the full model. However, by using WAIC$_Y$ we assume the chemical covariates are modeled appropriately.

4 Convergence Assessment

In all cases below, we show that convergence was adequately met within 5001 iterations. This convergence analysis was assessed using our RJAGS model.

In RJAGS, the program automatically adds a 1000 iteration adaptation period. This adaptation period is automatically discarded as part of the RJAGS sampling algorithm. In all analyses in this paper, we chose to use 25,000 iterations after 5000 iterations of burn.in. As previously noted 1,000 of these burn.in iterations are the adaptation period.

We define a model as satisfying convergence if Gelman Rubin diagnostics were low (below 1.1) and trace plots show strong mixing. We also observed the Monte Carlo Standard Error (MCSE) estimates, and confirmed they were reasonably low. We also provide assessments of autocorrelation below, including methods for avoiding autocorrelation for this type of model.

In the analysis of convergence below, please use the following key to identify parameters:

Parameter	Name
$\beta_{0,(1)}$	$beta[1]$
$\beta_{0,(2)}$	$beta[2]$
$\beta_{1,(2)}$	$beta[3]$
$\beta_{0,(Y\,\|\,\mathbf{X})}$	$beta[4]$
$\beta_{1,(Y\,\|\,\mathbf{X})}$	$beta[5]$
$\beta_{2,(Y\,\|\,\mathbf{X})}$	$beta[6]$
$\sigma^2_{(1)}$	$sigmax1$
$\sigma^2_{(2\|1)}$	$sigmax2x1$
$\sigma^2_{(Y\|\mathbf{X})}$	$sigmayx1x2$

Table 1: Parameter names for images

4.1 Simulation Data

We tested convergence on our simulated dataset with 25 percent censoring in $X_{(1)}$, $X_{(2)}$, and Y. We used 5001 iterations after the 1000 adaptation period to assess convergence. Thinning was not performed in this convergence check or in modeling of our data. We used three chains to assess convergence.

Gelman Rubin Diagnostics

Gelman Rubin statistics clearly indicated convergence with values well below 1.1.

Parameter	Gelman Rubin Statistic	Upper CI
$\beta_{0,(1)}$	1.00	1.00
$\beta_{0,(2)}$	1.00	1.01
$\beta_{1,(2)}$	1.00	1.01
$\beta_{0,(Y\,\vert\,\mathbf{X})}$	1.01	1.03
$\beta_{1,(Y\,\vert\,\mathbf{X})}$	1.00	1.00
$\beta_{2,(Y\,\vert\,\mathbf{X})}$	1.01	1.03
$\sigma^2_{(1)}$	1.00	1.00
$\sigma^2_{(2\vert1)}$	1.00	1.00
$\sigma^2_{(Y\vert\mathbf{X})}$	1.00	1.00

Table 2: Gelman Rubin statistics by parameter in the simulation study analysis. Statistics should be below 1.1 to indicate convergence.

MCSE:

All MCSE estimates were well below 0.01 (not shown). We consider the MCSE estimates to be appropriate if below 0.02.

Trace Plots:

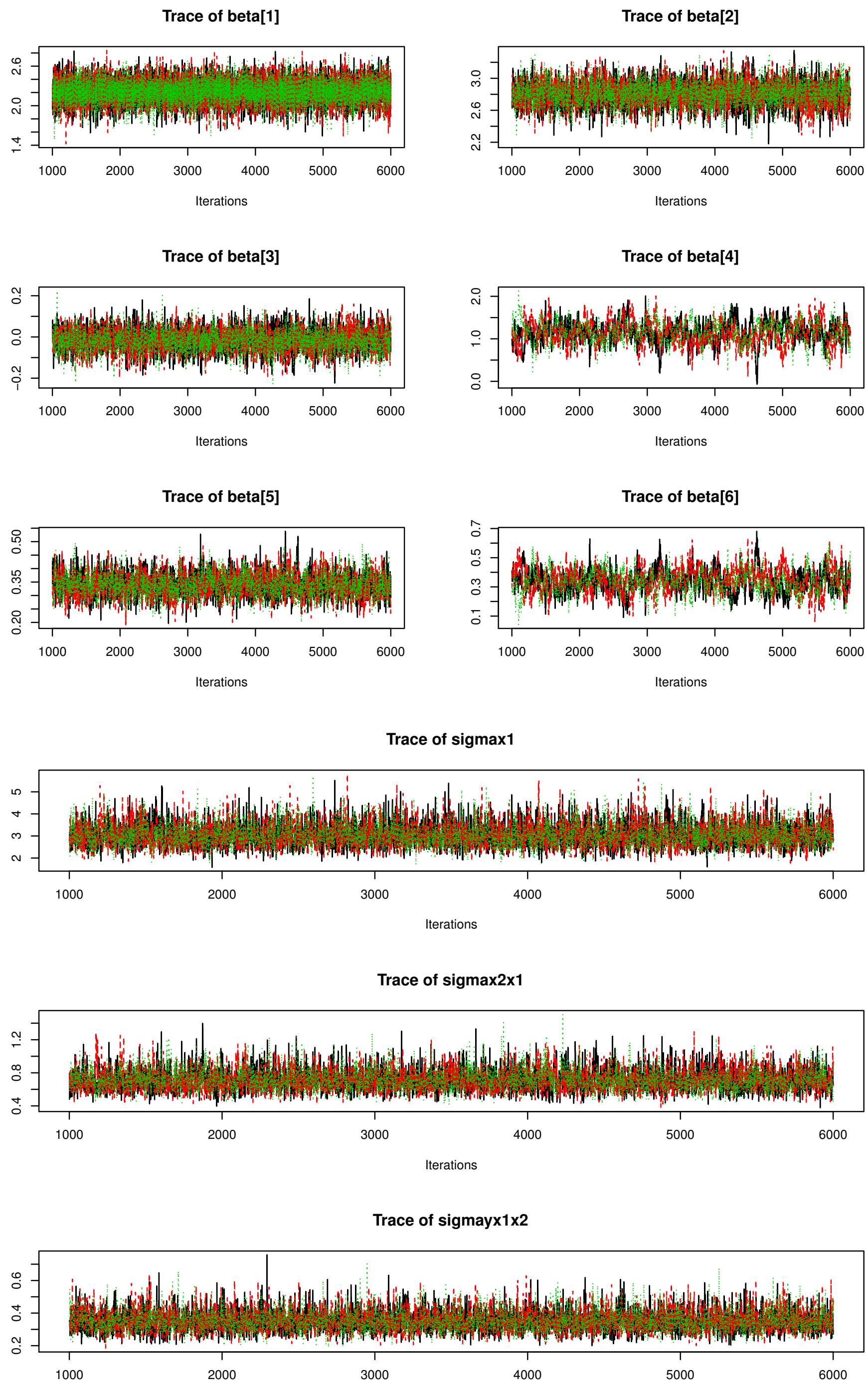

Figure 1: Trace plots for parameters in the simulation study. All parameters have appropriate mixing and convergence is evident.

Autocorrelation:

Autocorrelation was assessed through autocorrelation plots.

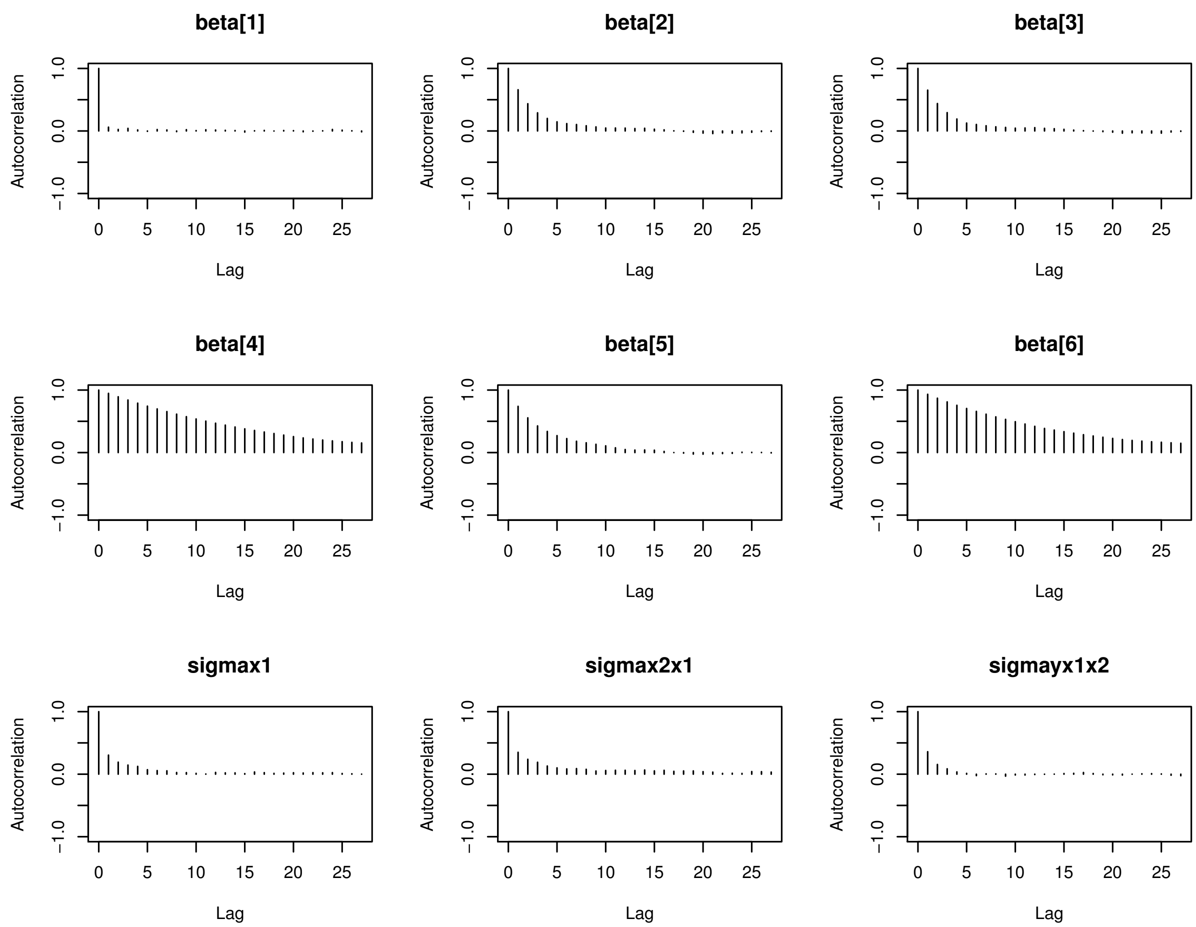

Figure 2: Autocorrelation plots by parameter for the simulation study.

These plots suggested some evidence of autocorrelation. Some autocorrelation may be present in these datasets because of the assumed linear relationships. When linear relationships are present and correlation is high, one may center the covariates to reduce autocorrelation.

To prove that centering will fix any autocorrelation detected or suspected here, we reran the model using centered covariates in the mean expressions. Centering was performed using the estimated mean at each iteration. The following figure of autocorrelation plots was obtained after centering the covariates.

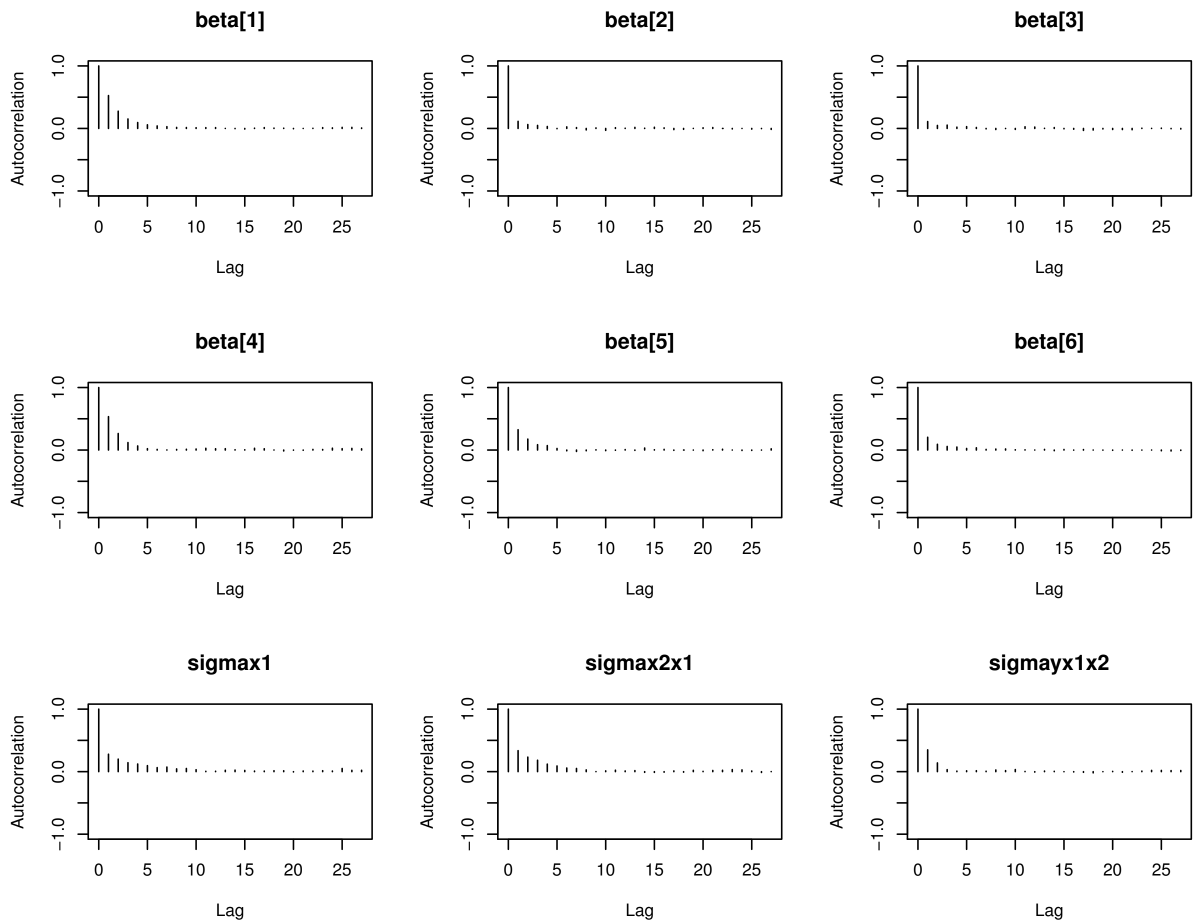

Figure 3: Autocorrelation plots by parameter for the simulation study using centered covariates.

From the above figure, it was clear that autocorrelation reduced when one centered the covariates in each conditional mean expression. If autocorrelation is extreme, we recommend centering the chemical covariates in each conditional mean expression.

In our simulation studies, we felt that centering was not necessary because strong autocorrelation was not present. In addition, since centering is a mathematical transformation, we expect all results to be similar if one uses a model with centered covariates in the mean expressions.

4.2 Real Data

We tested convergence on our real data set using 5001 iterations after 1000 iterations of adaptation. Thinning was not performed in this convergence check or in modeling of our data. We used three chains to assess convergence.

Gelman Rubin Diagnostics

Gelman Rubin statistics clearly indicated convergence.

Parameter	Gelman Rubin Statistic	Upper CI
$\beta_{0,(1)}$	1.00	1.00
$\beta_{0,(2)}$	1.00	1.00
$\beta_{1,(2)}$	1.00	1.00
$\beta_{0,(Y\,\mid\,\mathbf{X})}$	1.00	1.01
$\beta_{1,(Y\,\mid\,\mathbf{X})}$	1.00	1.00
$\beta_{2,(Y\,\mid\,\mathbf{X})}$	1.00	1.01
$\sigma^2_{(1)}$	1.00	1.00
$\sigma^2_{(2\|1)}$	1.00	1.02
$\sigma^2_{(Y\|\mathbf{X})}$	1.00	1.00

Table 3: Gelman Rubin statistics by parameter in the real data analysis. Statistics should be below 1.1 to indicate convergence.

MCSE:

All MCSE estimates were well below 0.02 (not shown). Most were below 0.01. We consider the MCSE estimates to be appropriate if below 0.02.

Trace Plots:

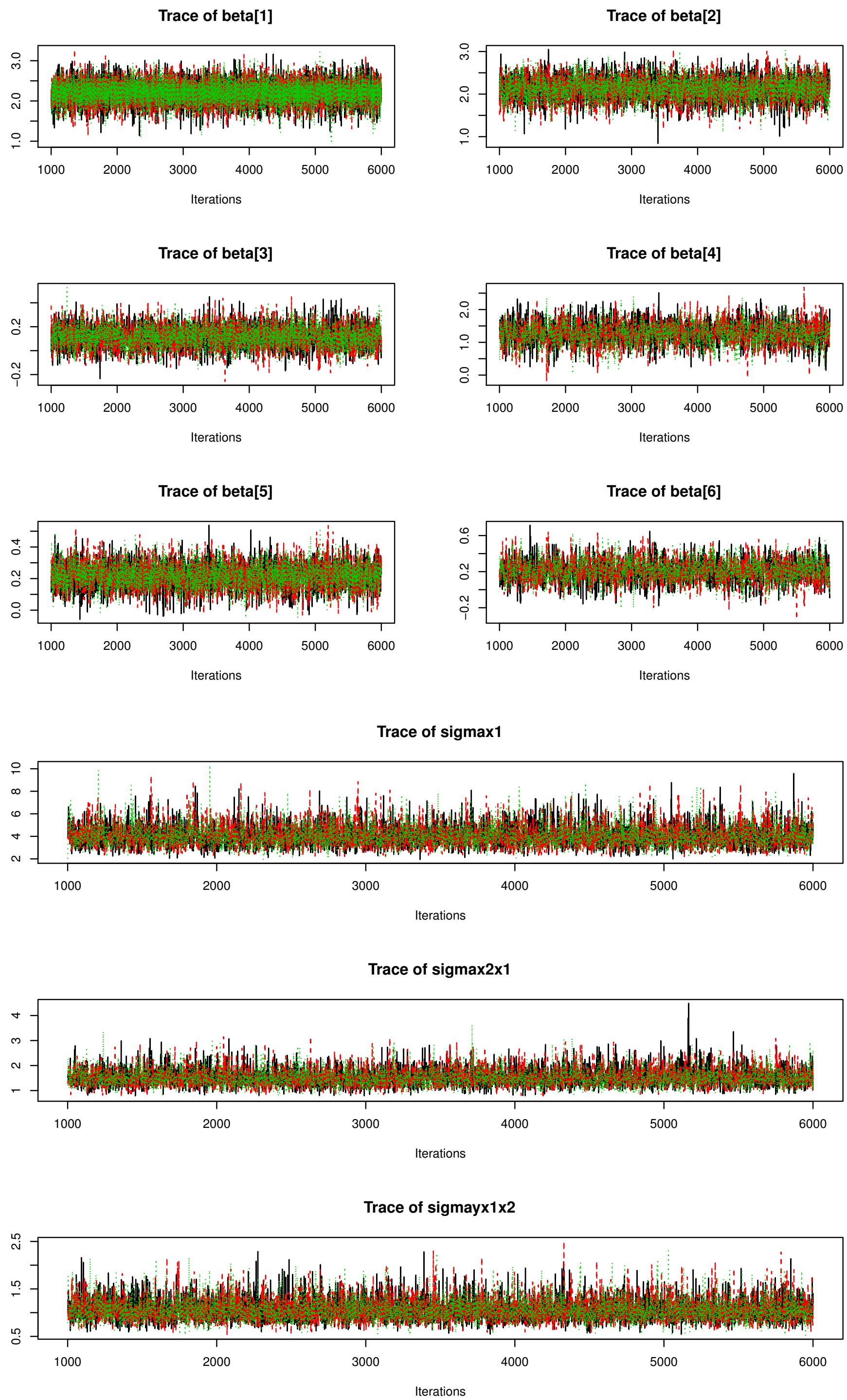

Figure 4: Trace plots for parameters in the real data. All parameters have appropriate mixing and convergence is evident.

Autocorrelation:

Autocorrelation was assessed through autocorrelation plots. Autocorrelation appeared to be minor in the real data. As described in the simulation, centering the chemical covariates would eliminate any autocorrelation concerns. Thinning was not applied to the real data.

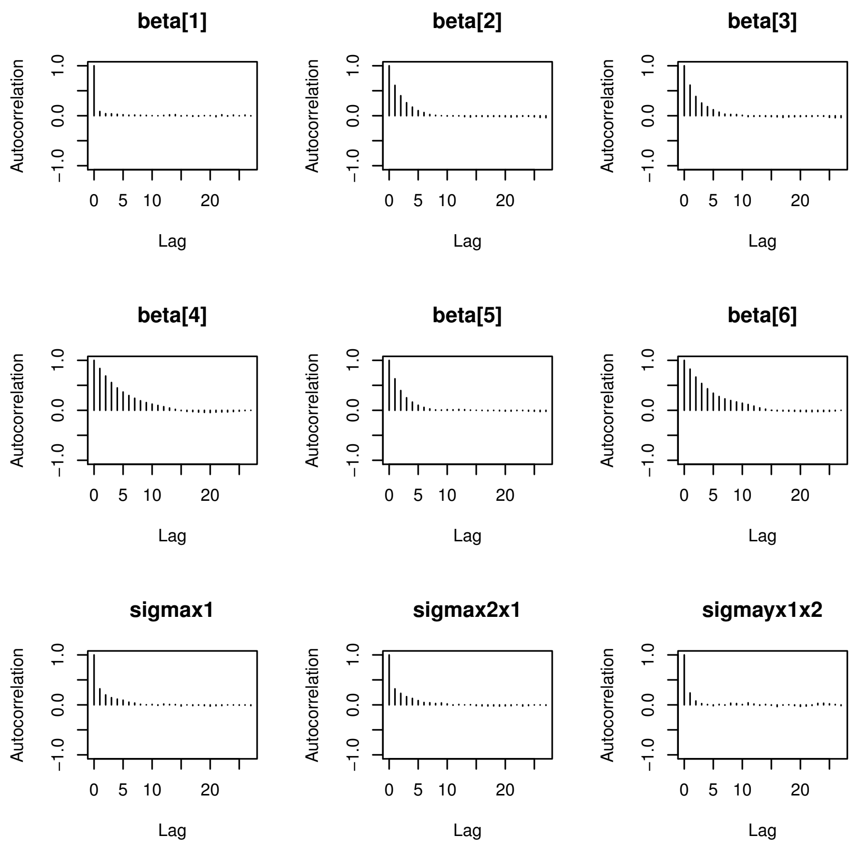

Figure 5: Autocorrelation plots by parameter for the real data.

5 Simulation Studies

5.1 Model Comparison: WAIC$_Y$

To compare how well we could model Y, we compared a series of four models with WAIC$_Y$. The first model was the true model described above (used for data generation in section 4.1) where the response Y depended on both $X_{(1)}$ and $X_{(2)}$. The second model was a bivariate model that assumes that our response Y was dependent on only $X_{(1)}$. Our third model was a bivariate model where we assumed our response Y was dependent on only $X_{(2)}$. Then, finally, our last model used an ANOVA framework (intercept only regression model) and did not assume any dependence on $X_{(1)}$ or $X_{(2)}$.

To compare models, we calculated WAIC$_Y$ for each model in each modeling scenario. In order to assess variability in WAIC$_Y$, we also calculated WAIC$_Y$ for 100 runs under each model with different model seeds (but no change to the raw dataset). We also calculated P_1 to identify if trends differed with the penalty term used.

Table 4 shows the results of the WAIC$_Y$ model comparison.

In the non-censored scenario (percent censoring of 0), the 3-variable model had the lowest WAIC$_Y$, indicating this model would be preferred over simpler models. This result confirms that our dataset was generated appropriately and under normal regression scenarios. This result was expected because the 3-variable model is the true model, i.e. the model under which the data were generated.

Across all level of censoring in Y, $X_{(1)}$, and $X_{(2)}$, WAIC$_Y$ was lowest for the 3-variable model indicating that the 3-variable model would be preferred to other simpler models in all scenarios. The ANOVA model had the highest WAIC$_Y$ across each scenario. This indicated that models including a chemical covariate were preferred over simply modeling the mean exposure. The bivariate model with $X_{(1)}$ had the second lowest WAIC$_Y$ values, indicating that $X_{(1)}$ explained slightly more variation than $X_{(2)}$, also as expected. Variability in WAIC$_Y$

Percent Censoring			WAIC_Y			
			3-Variable	Bivariate Model	Bivariate Model	ANOVA
$X_{(1)}$	$X_{(2)}$	Y	Model	with X_1	with X_2	Model
0	0	0	179.7	196.7	247.7	254.6
25	25	25	178.0	193.6	237.2	243.7
25	25	50	152.2	166.6	212.4	218.0
25	25	75	107.1	116.3	152.4	155.1
25	50	25	178.0	193.6	238.6	243.7
25	50	50	152.4	166.6	213.9	218.0
25	50	75	106.1	116.3	153.1	155.1
25	75	25	184.9	193.6	240.6	243.7
25	75	50	155.6	166.6	214.3	218.0
25	75	75	108.4	116.3	153.5	155.1
50	25	25	185.0	199.2	237.2	243.7
50	25	50	154.4	166.8	212.4	218.0
50	25	75	111.4	119.5	152.4	155.1
50	50	25	185.4	199.2	238.6	243.7
50	50	50	154.4	166.8	213.9	218.0
50	50	75	110.6	119.5	153.1	155.1
50	75	25	191.6	199.2	240.6	243.7
50	75	50	157.5	166.8	214.3	218.0
50	75	75	112.2	119.5	153.5	155.1
75	25	25	187.3	198.7	237.2	243.7
75	25	50	162.3	172.0	212.4	218.0
75	25	75	106.5	114.1	152.4	155.1
75	50	25	186.2	198.7	238.6	243.7
75	50	50	161.8	172.0	213.9	218.0
75	50	75	105.6	114.1	153.1	155.1
75	75	25	189.4	198.7	240.6	243.7
75	75	50	162.3	172.0	214.3	218.0
75	75	75	106.1	114.1	153.5	155.1

Table 4: WAIC_Y statistics by model type for various different datasets with different degrees of censoring in $X_{(1)}$,$X_{(2)}$, and Y. WAIC_Y values are reported using the same model seed.

was minimal, and trends were similar for WAIC_Y calculated with penalty P_1.

5.2 WAIC$_Y$ Variability Results

Results assessing the variability in WAIC$_Y$ are included in tables 5-8.

Percent Censoring			3-Variable Model									
			$LPPD$		P_2		WAIC$_Y$ with P_2		P_1		WAIC$_Y$ with P_1	
$X_{(1)}$	$X_{(2)}$	Y	5%	95%	5%	95%	5%	95%	5%	95%	5%	95%
0	0	0	-85.4	-85.4	4.4	4.5	179.6	179.9	4.1	4.3	179.1	179.3
25	25	25	-81.6	-81.5	7.4	7.5	177.7	178.1	6.6	6.7	176.2	176.5
25	25	50	-70.8	-70.7	5.3	5.5	152.1	152.5	4.8	4.9	151.0	151.4
25	25	75	-49.6	-49.5	4.0	4.2	107.0	107.6	3.6	3.8	106.1	106.6
25	50	25	-79.5	-79.3	9.5	9.7	177.7	178.0	8.4	8.6	175.5	175.9
25	50	50	-69.1	-69.0	7.0	7.2	152.1	152.5	6.2	6.4	150.5	150.9
25	50	75	-48.3	-48.2	4.7	4.9	105.9	106.3	4.1	4.3	104.7	105.1
25	75	25	-81.3	-81.1	11.2	11.4	184.7	185.1	9.9	10.1	182.2	182.6
25	75	50	-68.1	-67.9	9.7	9.9	155.3	155.7	8.4	8.5	152.8	153.1
25	75	75	-47.7	-47.4	6.4	6.7	107.9	108.5	5.5	5.7	106.0	106.5
50	25	25	-79.4	-79.2	13.2	13.3	184.8	185.3	11.0	11.1	180.5	180.9
50	25	50	-67.4	-67.2	9.7	9.9	153.9	154.5	7.9	8.1	150.4	150.8
50	25	75	-49.5	-49.3	6.2	6.4	111.1	111.8	5.0	5.2	108.7	109.3
50	50	25	-78.3	-78.1	14.4	14.6	185.2	185.7	11.9	12.1	180.2	180.7
50	50	50	-66.3	-66.1	11.0	11.2	154.4	154.9	9.0	9.1	150.3	150.7
50	50	75	-48.4	-48.3	6.9	7.1	110.4	111.0	5.6	5.7	107.7	108.2
50	75	25	-79.8	-79.6	16.0	16.2	191.3	191.8	13.3	13.5	186.0	186.5
50	75	50	-65.8	-65.6	12.8	13.0	157.0	157.5	10.5	10.6	152.2	152.7
50	75	75	-48.0	-47.8	8.2	8.4	112.2	112.7	6.6	6.8	109.0	109.4
75	25	25	-70.6	-70.3	23.0	23.2	186.8	187.5	17.9	18.0	176.4	177.1
75	25	50	-63.0	-62.7	18.0	18.3	161.7	162.5	13.9	14.1	153.4	154.2
75	25	75	-43.8	-43.5	9.6	9.8	106.4	107.1	7.3	7.5	101.9	102.5
75	50	25	-68.5	-68.2	24.5	24.8	185.8	186.5	18.8	19.0	174.2	174.9
75	50	50	-61.4	-61.1	19.5	19.7	161.3	162.1	15.0	15.1	152.1	152.9
75	50	75	-42.3	-42.1	10.4	10.6	105.0	105.8	7.9	8.1	100.1	100.7
75	75	25	-68.7	-68.2	26.3	26.5	189.4	190.2	20.1	20.2	176.8	177.7
75	75	50	-60.2	-59.8	21.1	21.3	162.0	162.8	16.1	16.2	151.9	152.6
75	75	75	-41.3	-41.0	11.7	11.9	105.4	106.2	8.9	9.1	99.9	100.6

Table 5: WAIC$_Y$ statistic 5 and 95 quantiles for the 3-variable model for various different datasets with different degrees of censoring in $X_{(1)}$,$X_{(2)}$, and Y.

Percent Censoring			Bivariate with $X_{(1)}$									
			$LPPD$		P_2		WAIC$_Y$ with P_2		P_1		WAIC$_Y$ with P_1	
$X_{(1)}$	$X_{(2)}$	Y	5%	95%	5%	95%	5%	95%	5%	95%	5%	95%
0	-	0	-94.7	-94.7	3.6	3.7	196.6	196.8	3.4	3.5	196.2	196.4
25	-	25	-91.6	-91.6	5.2	5.3	193.5	193.7	4.7	4.8	192.6	192.8
25	-	50	-79.7	-79.6	3.7	3.8	166.7	166.9	3.4	3.5	166.1	166.3
25	-	75	-55.8	-55.7	2.4	2.6	116.2	116.6	2.3	2.4	116.0	116.3
50	-	25	-88.0	-87.9	11.6	11.7	199.1	199.4	9.8	9.9	195.5	195.8
50	-	50	-74.5	-74.4	8.9	9.1	166.8	167.2	7.3	7.4	163.5	163.9
50	-	75	-54.8	-54.7	5.0	5.2	119.4	119.8	4.1	4.2	117.6	117.9
75	-	25	-76.9	-76.6	22.5	22.7	198.5	199.0	17.5	17.7	188.4	189.0
75	-	50	-68.1	-67.8	18.2	18.4	172.1	172.7	14.0	14.1	163.7	164.3
75	-	75	-47.8	-47.6	9.2	9.3	113.7	114.2	7.0	7.1	109.2	109.7

Table 6: WAIC$_Y$ statistic 5 and 95 quantiles for the bivariate model with $X_{(1)}$ for various different datasets with different degrees of censoring in $X_{(1)}$,$X_{(2)}$, and Y.

Percent Censoring			Bivariate with $X_{(2)}$									
			$LPPD$		P_2		WAIC$_Y$ with P_2		P_1		WAIC$_Y$ with P_1	
$X_{(1)}$	$X_{(2)}$	Y	5%	95%	5%	95%	5%	95%	5%	95%	5%	95%
-	0	0	-120.3	-120.3	3.5	3.6	247.6	247.8	3.3	3.4	247.2	247.4
-	25	25	-114.9	-114.8	3.8	3.9	237.2	237.4	3.6	3.7	236.8	237.1
-	25	50	-102.8	-102.8	3.4	3.5	212.3	212.6	3.2	3.3	211.9	212.2
-	25	75	-73.3	-73.2	2.8	2.9	152.1	152.5	2.6	2.7	151.7	152.1
-	50	25	-114.1	-114.0	5.1	5.3	238.4	238.7	4.9	5.0	237.9	238.2
-	50	50	-102.5	-102.4	4.4	4.5	213.8	214.0	4.1	4.2	213.2	213.4
-	50	75	-73.1	-73.0	3.4	3.6	152.9	153.3	3.1	3.3	152.3	152.7
-	75	25	-113.9	-113.7	6.5	6.7	240.5	240.7	6.1	6.3	239.9	240.0
-	75	50	-100.9	-100.6	6.3	6.5	214.3	214.5	5.8	6.0	213.2	213.5
-	75	75	-72.2	-72.0	4.6	4.7	153.4	153.7	4.1	4.3	152.5	152.8

Table 7: WAIC$_Y$ statistic 5 and 95 quantiles for the bivariate model with $X_{(2)}$ for various different datasets with different degrees of censoring in $X_{(1)}$,$X_{(2)}$, and Y.

Percent Censoring			ANOVA Model									
			$LPPD$		P_2		WAIC$_Y$ with P_2		P_1		WAIC$_Y$ with P_1	
$X_{(1)}$	$X_{(2)}$	Y	5%	95%	5%	95%	5%	95%	5%	95%	5%	95%
-	-	0	-124.6	-124.6	2.7	2.7	254.6	254.7	2.5	2.6	254.3	254.4
-	-	25	-119.6	-119.5	2.3	2.4	243.7	243.8	2.2	2.3	243.5	243.6
-	-	50	-106.9	-106.9	2.1	2.1	217.9	218.1	2.0	2.0	217.8	217.9
-	-	75	-75.9	-75.9	1.6	1.7	155.0	155.2	1.6	1.7	154.9	155.1

Table 8: WAIC$_Y$ statistic 5 and 95 quantiles for the ANOVA model for various different datasets with different degrees of censoring in $X_{(1)}$,$X_{(2)}$, and Y.

5.3 Multicollinearity Simulation

In Tables 9-12, we report the coverage probabilities for the regression parameters (note $\beta_{1,(2)}$ will be defined differently based on the correlation level) and the conditional variances. We also report median posterior 95% credible interval (CI) width of the regression parameters.

Results of this simulation suggest that multicollinearity will be present in our models. We see an inflation in the CI width for the regression parameters $\beta_{1,(Y\,|\,\mathbf{X})}$ and $\beta_{2,(Y\,|\,\mathbf{X})}$ which are the coefficients that correspond to the slopes of $X_{(1)}$ and $X_{(2)}$ for Y. When the correlation is set to 0.75, the CI widths are double or triple what they were under a correlation of 0.

However, this result does not seem dependent on censoring, as similar increases in CI width were seen for the non-censored sets.

Limited differences in coverage were identified between different correlation levels. The inflation of the CI width estimates did not appear to influence the patterns in the coverage probabilities that occur with increased censoring.

Percent Censoring			95% Coverage Probabilities									Median Posterior 95% CI Width					
			Regression Coefficients						Variances			Regression Coefficients					
Y	$X_{(1)}$	$X_{(2)}$	$\beta_{0,(1)}$	$\beta_{0,(2)}$	$\beta_{1,(2)}$	$\beta_{0,(Y\|\mathbf{X})}$	$\beta_{1,(Y\|\mathbf{X})}$	$\beta_{2,(Y\|\mathbf{X})}$	$\sigma^2_{(1)}$	$\sigma^2_{(2\|1)}$	$\sigma^2_{(Y\|\mathbf{X})}$	$\beta_{0,(1)}$	$\beta_{0,(2)}$	$\beta_{1,(2)}$	$\beta_{0,(Y\|\mathbf{X})}$	$\beta_{1,(Y\|\mathbf{X})}$	$\beta_{2,(Y\|\mathbf{X})}$
0	0	0	94.3	95.8	95.3	94.2	94.6	94.3	95.2	94.9	94.0	0.59	0.64	0.22	0.94	0.16	0.29
25	25	25	92.6	95.8	94.9	95.7	95.2	95.4	83.0	90.2	91.7	0.68	0.66	0.22	1.12	0.19	0.33
25	25	50	92.6	89.4	95.5	94.5	95.9	94.6	82.9	47.1	93.0	0.68	0.86	0.28	1.04	0.19	0.31
25	25	75	92.6	40.5	95.5	74.1	95.8	80.0	83.2	1.1	94.6	0.68	1.62	0.47	0.90	0.19	0.27
25	50	25	62.7	95.2	95.2	92.9	87.1	95.9	33.5	89.9	93.8	0.91	0.58	0.20	1.13	0.18	0.34
25	50	50	61.9	86.5	95.2	84.6	86.9	94.5	33.8	48.3	94.0	0.91	0.76	0.25	1.05	0.18	0.32
25	50	75	62.3	27.3	95.9	44.8	86.6	81.2	33.9	1.5	94.9	0.91	1.48	0.42	0.91	0.19	0.28
25	75	25	1.0	95.5	95.2	62.9	40.4	95.1	1.9	91.2	95.8	1.85	0.45	0.18	1.21	0.18	0.37
25	75	50	1.0	78.9	95.0	43.1	39.4	95.6	1.9	50.1	95.0	1.85	0.60	0.22	1.13	0.18	0.35
25	75	75	1.1	7.7	94.1	8.8	41.5	83.2	1.7	1.8	95.3	1.85	1.26	0.36	0.96	0.18	0.31
50	25	25	92.7	95.7	95.2	79.1	82.5	91.8	81.2	89.7	65.5	0.68	0.66	0.22	1.56	0.25	0.43
50	25	50	92.4	89.4	95.6	88.1	82.2	94.8	81.2	46.7	68.9	0.68	0.86	0.28	1.46	0.26	0.40
50	25	75	92.7	39.8	95.1	97.2	84.2	92.6	82.1	1.2	73.5	0.68	1.63	0.47	1.26	0.26	0.35
50	50	25	58.7	94.9	95.6	92.7	96.4	92.5	29.7	89.9	72.4	0.94	0.57	0.20	1.54	0.24	0.43
50	50	50	58.8	85.7	94.3	95.8	97.1	95.7	29.5	47.7	74.8	0.94	0.75	0.25	1.44	0.24	0.40
50	50	75	58.4	25.1	96.3	94.2	96.1	92.3	30.0	1.6	79.7	0.94	1.48	0.42	1.23	0.25	0.35
50	75	25	0.8	95.0	95.1	94.5	84.9	92.4	1.0	90.9	84.6	1.94	0.44	0.17	1.56	0.22	0.46
50	75	50	0.9	77.5	94.9	87.5	84.6	94.2	0.9	50.4	86.2	1.94	0.59	0.22	1.45	0.23	0.43
50	75	75	0.6	4.9	94.8	54.1	85.0	91.7	0.9	1.8	89.2	1.94	1.26	0.36	1.22	0.23	0.37
75	25	25	92.5	95.7	94.9	23.0	40.2	83.8	81.7	89.8	16.8	0.68	0.66	0.22	3.18	0.48	0.73
75	25	50	92.6	89.4	95.1	27.7	40.3	88.3	82.1	47.0	17.9	0.68	0.86	0.28	3.02	0.48	0.68
75	25	75	92.6	39.1	95.1	43.6	43.0	95.5	81.8	1.1	24.3	0.68	1.65	0.47	2.62	0.49	0.58
75	50	25	56.5	95.4	95.7	38.6	59.3	84.3	29.1	90.0	22.2	0.96	0.57	0.20	3.11	0.45	0.74
75	50	50	56.7	85.7	94.8	44.8	59.7	88.6	29.0	46.7	24.0	0.96	0.75	0.25	2.92	0.46	0.69
75	50	75	56.4	24.6	95.8	68.3	61.7	95.6	29.3	1.1	30.5	0.96	1.49	0.43	2.52	0.46	0.58
75	75	25	0.4	95.1	95.0	81.9	95.4	85.6	0.5	90.8	37.5	2.04	0.43	0.17	2.94	0.40	0.77
75	75	50	0.4	77.9	95.3	88.6	95.2	89.9	0.6	50.2	40.2	2.04	0.58	0.22	2.74	0.39	0.72
75	75	75	0.3	4.4	94.6	98.4	95.8	96.2	0.5	1.8	47.7	2.05	1.26	0.36	2.32	0.40	0.61

Table 9: Coverage probabilities and median 95% CI width for different degrees of censoring in $X_{(1)}$,$X_{(2)}$, and Y when the correlation between $X_{(1)}$ and $X_{(2)}$ is 0.00

Percent Censoring			95% Coverage Probabilities									Median Posterior 95% CI Width					
			Regression Coefficients						Variances			Regression Coefficients					
Y	$X_{(1)}$	$X_{(2)}$	$\beta_{0,(1)}$	$\beta_{0,(2)}$	$\beta_{1,(2)}$	$\beta_{0,(Y\|\mathbf{X})}$	$\beta_{1,(Y\|\mathbf{X})}$	$\beta_{2,(Y\|\mathbf{X})}$	$\sigma^2_{(1)}$	$\sigma^2_{(2\|1)}$	$\sigma^2_{(Y\|\mathbf{X})}$	$\beta_{0,(1)}$	$\beta_{0,(2)}$	$\beta_{1,(2)}$	$\beta_{0,(Y\|\mathbf{X})}$	$\beta_{1,(Y\|\mathbf{X})}$	$\beta_{2,(Y\|\mathbf{X})}$
0	0	0	94.3	95.8	95.3	93.9	93.9	93.9	95.2	94.9	94.0	0.59	0.64	0.21	0.93	0.21	0.29
25	25	25	92.5	96.4	95.5	95.7	95.1	95.5	80.7	91.3	91.7	0.68	0.79	0.25	1.16	0.25	0.34
25	25	50	92.7	63.4	78.3	93.4	94.8	95.4	79.4	58.4	92.8	0.69	1.17	0.34	1.07	0.26	0.33
25	25	75	92.4	4.1	33.3	57.7	94.2	87.3	80.8	13.6	93.7	0.68	2.47	0.60	0.92	0.30	0.31
25	50	25	64.8	87.6	92.0	91.0	92.4	95.4	34.1	93.0	92.9	0.89	0.79	0.24	1.25	0.24	0.36
25	50	50	62.4	94.9	94.6	79.4	89.9	93.3	31.2	67.3	93.9	0.91	1.11	0.32	1.15	0.25	0.34
25	50	75	61.5	17.2	56.9	28.8	89.1	85.0	30.9	19.0	94.1	0.91	2.33	0.57	0.95	0.28	0.32
25	75	25	2.6	16.4	47.9	69.6	70.6	95.3	3.4	94.4	95.0	1.68	0.84	0.24	1.54	0.24	0.41
25	75	50	2.3	83.5	91.3	46.0	69.0	92.2	2.6	77.4	95.3	1.72	1.08	0.31	1.42	0.24	0.39
25	75	75	1.9	92.8	95.5	8.1	66.9	83.4	2.1	30.3	95.4	1.76	2.06	0.51	1.14	0.27	0.35
50	25	25	92.5	96.6	96.1	78.7	89.2	91.9	79.6	91.0	68.6	0.69	0.79	0.25	1.62	0.32	0.43
50	25	50	92.4	61.8	76.0	87.9	92.3	94.0	77.7	57.0	71.3	0.69	1.18	0.34	1.51	0.33	0.41
50	25	75	92.3	3.1	29.4	98.1	94.5	93.8	78.6	13.0	77.8	0.69	2.54	0.62	1.29	0.38	0.38
50	50	25	60.4	85.8	91.0	92.6	94.6	94.9	28.8	93.2	74.6	0.92	0.79	0.24	1.69	0.31	0.45
50	50	50	57.5	95.4	94.4	96.3	96.0	95.5	24.9	66.8	76.9	0.95	1.13	0.32	1.57	0.32	0.43
50	50	75	55.4	15.8	53.3	93.0	96.3	92.7	24.9	18.7	81.4	0.96	2.40	0.58	1.29	0.36	0.39
50	75	25	1.7	12.2	42.6	93.7	93.5	95.3	1.5	94.9	83.5	1.77	0.83	0.24	1.97	0.29	0.50
50	75	50	1.1	79.7	90.3	87.9	91.1	95.2	1.2	80.4	85.3	1.83	1.09	0.30	1.82	0.30	0.48
50	75	75	1.0	93.9	95.7	54.3	89.6	91.2	0.4	30.9	88.8	1.88	2.09	0.51	1.46	0.33	0.43
75	25	25	92.6	96.4	96.1	21.6	65.2	86.5	80.4	91.0	23.4	0.69	0.79	0.25	3.25	0.56	0.72
75	25	50	92.5	61.6	77.2	24.9	72.7	88.2	79.6	56.3	26.7	0.69	1.19	0.35	3.09	0.57	0.69
75	25	75	92.3	3.0	29.4	48.2	80.6	92.6	80.2	12.1	34.4	0.69	2.61	0.63	2.65	0.62	0.62
75	40	25	58.9	84.5	89.7	42.0	74.0	88.4	28.9	93.5	27.5	0.93	0.79	0.24	3.29	0.54	0.74
75	50	50	55.8	96.4	95.5	44.9	78.9	90.3	26.0	66.4	30.4	0.96	1.13	0.33	3.11	0.55	0.70
75	50	75	53.1	14.6	50.7	72.2	84.8	93.8	23.9	17.0	39.0	0.97	2.47	0.60	2.65	0.59	0.62
75	75	25	0.8	8.7	37.3	85.3	92.8	93.6	1.5	94.9	41.8	1.86	0.83	0.24	3.46	0.49	0.79
75	75	50	0.5	74.3	88.5	90.1	93.8	94.3	0.6	82.5	43.2	1.95	1.09	0.30	3.27	0.50	0.77
75	75	75	0.3	94.7	95.7	97.8	95.2	95.1	0.4	29.8	48.9	2.04	2.12	0.51	2.68	0.53	0.67

Table 10: Coverage probabilities and median 95% CI width for different degrees of censoring in $X_{(1)}$,$X_{(2)}$, and Y when the correlation between $X_{(1)}$ and $X_{(2)}$ is 0.25

Percent Censoring			95% Coverage Probabilities									Median Posterior 95% CI Width					
			Regression Coefficients						Variances			Regression Coefficients					
Y	$X_{(1)}$	$X_{(2)}$	$\beta_{0,(1)}$	$\beta_{0,(2)}$	$\beta_{1,(2)}$	$\beta_{0,(Y\|\mathbf{X})}$	$\beta_{1,(Y\|\mathbf{X})}$	$\beta_{2,(Y\|\mathbf{X})}$	$\sigma^2_{(1)}$	$\sigma^2_{(2\|1)}$	$\sigma^2_{(Y\|\mathbf{X})}$	$\beta_{0,(1)}$	$\beta_{0,(2)}$	$\beta_{1,(2)}$	$\beta_{0,(Y\|\mathbf{X})}$	$\beta_{1,(Y\|\mathbf{X})}$	$\beta_{2,(Y\|\mathbf{X})}$
0	0	0	94.3	95.8	95.3	94.5	95.8	94.6	95.2	94.9	94.0	0.59	0.64	0.22	0.95	0.32	0.30
25	25	25	92.7	96.0	95.2	95.7	95.6	95.5	76.2	94.2	91.7	0.69	0.92	0.28	1.20	0.38	0.35
25	25	50	91.9	52.1	66.2	92.8	95.3	94.8	75.3	75.6	93.0	0.69	1.38	0.39	1.10	0.41	0.35
25	25	75	92.3	2.4	13.3	58.2	95.0	92.3	79.0	39.1	93.9	0.69	2.87	0.68	0.99	0.50	0.37
25	50	25	70.4	82.4	88.3	93.0	95.9	95.3	37.3	94.5	91.8	0.85	1.04	0.30	1.37	0.39	0.38
25	50	50	60.9	96.4	95.6	77.2	94.5	93.8	26.9	84.0	94.0	0.90	1.43	0.39	1.25	0.41	0.38
25	50	75	60.6	17.3	38.6	27.4	91.7	91.8	28.8	48.0	95.1	0.91	2.85	0.68	1.05	0.49	0.37
25	75	25	9.7	11.5	32.6	83.2	90.8	94.2	6.8	94.3	94.3	1.39	1.35	0.34	1.85	0.40	0.46
25	75	50	4.9	68.7	84.0	58.3	88.9	92.1	2.6	91.5	95.2	1.52	1.62	0.41	1.71	0.42	0.44
25	75	75	3.0	97.3	96.8	13.6	86.9	87.5	1.7	69.5	96.3	1.68	2.78	0.65	1.38	0.48	0.42
50	25	25	92.10	96.1	95.4	76.4	92.9	93.3	75.5	93.8	73.1	0.69	0.93	0.29	1.66	0.47	0.43
50	25	50	91.4	46.4	62.7	86.8	95.0	93.1	74.00	72.1	75.1	0.70	1.42	0.40	1.55	0.51	0.43
50	25	75	92.1	1.5	9.9	98.4	94.9	94.7	77.2	36.0	81.3	0.70	2.98	0.71	1.37	0.61	0.44
50	50	25	66.6	78.4	85.1	92.4	93.3	95.2	29.8	94.2	76.9	0.88	1.06	0.30	1.81	0.47	0.46
50	50	50	52.3	96.4	96.0	96.9	95.6	94.6	19.0	83.5	79.7	0.96	1.46	0.40	1.67	0.50	0.46
50	50	75	48.9	14.5	34.2	90.6	95.2	94.8	20.7	45.3	84.3	0.98	2.96	0.70	1.39	0.59	0.45
50	75	25	7.7	7.5	26.8	95.6	96.0	95.1	4.2	94.1	83.2	1.45	1.34	0.35	2.34	0.48	0.56
50	75	50	2.2	64.3	82.5	91.3	95.3	93.2	0.8	92.5	86.4	1.63	1.64	0.41	2.18	0.50	0.54
50	75	75	0.9	97.4	96.2	55.6	93.2	91.9	0.3	69.1	90.9	1.85	2.83	0.66	1.71	0.57	0.51
75	25	25	92.1	96.1	95.9	20.8	84.2	87.9	77.0	93.6	33.8	0.69	0.93	0.29	3.30	0.78	0.71
75	25	50	91.7	45.6	61.6	20.3	88.9	87.2	76.4	71.90	37.4	0.70	1.43	0.40	3.14	0.80	0.69
75	25	75	92.3	1.1	8.8	51.1	92.9	91.1	79.6	34.5	46.4	0.69	3.10	0.74	2.77	0.91	0.67
75	50	25	65.0	76.9	84.3	38.6	84.1	92.1	31.1	94.9	37.3	0.88	1.06	0.30	3.36	0.76	0.72
75	50	50	49.0	96.8	95.9	44.3	88.3	90.7	19.1	82.5	41.1	0.98	1.48	0.40	3.21	0.78	0.71
75	50	75	45.7	11.2	29.6	73.9	94.1	92.8	19.9	42.70	49.6	1.00	3.09	0.74	2.80	0.89	0.68
75	75	25	5.8	5.3	21.3	82.8	88.7	94.8	3.3	94.4	47.1	1.50	1.35	0.35	3.85	0.74	0.82
75	75	50	1.3	55.0	77.4	88.3	91.4	94.6	0.6	92.1	50.4	1.75	1.65	0.42	3.69	0.75	0.81
75	75	75	0.4	97.5	96.3	97.9	94.4	95.3	0.1	67.6	61.0	2.08	2.92	0.67	2.98	0.85	0.75

Table 11: Coverage probabilities and median 95% CI width for different degrees of censoring in $X_{(1)}$,$X_{(2)}$, and Y when the correlation between $X_{(1)}$ and $X_{(2)}$ is 0.50

Percent Censoring			95% Coverage Probabilities									Median Posterior 95% CI Width					
			Regression Coefficients						Variances			Regression Coefficients					
Y	$X_{(1)}$	$X_{(2)}$	$\beta_{0,(1)}$	$\beta_{0,(2)}$	$\beta_{1,(2)}$	$\beta_{0,(Y\|\mathbf{X})}$	$\beta_{1,(Y\|\mathbf{X})}$	$\beta_{2,(Y\|\mathbf{X})}$	$\sigma^2_{(1)}$	$\sigma^2_{(2\|1)}$	$\sigma^2_{(Y\|\mathbf{X})}$	$\beta_{0,(1)}$	$\beta_{0,(2)}$	$\beta_{1,(2)}$	$\beta_{0,(Y\|\mathbf{X})}$	$\beta_{1,(Y\|\mathbf{X})}$	$\beta_{2,(Y\|\mathbf{X})}$
0	0	0	94.3	95.8	95.3	95.0	96.3	95.5	95.2	94.9	94.0	0.59	0.64	0.22	0.96	0.46	0.30
25	25	25	92.2	95.7	95.3	95.7	95.2	94.7	72.3	94.4	91.7	0.69	1.00	0.30	1.22	0.54	0.35
25	25	50	92.1	50.7	61.9	92.7	95.2	95.7	73.3	84.6	93.2	0.70	1.48	0.41	1.14	0.59	0.37
25	25	75	92.3	2.9	11.9	60.6	95.3	94.2	78.0	61.8	94.5	0.69	3.02	0.71	1.06	0.74	0.40
25	50	25	77.2	81.8	86.1	93.2	95.9	94.9	40.8	94.6	91.7	0.81	1.22	0.34	1.45	0.56	0.40
25	50	50	59.7	96.3	95.6	75.3	94.6	93.6	23.3	90.6	93.2	0.90	1.59	0.43	1.32	0.60	0.40
25	50	75	62.4	22.0	40.0	28.3	93.3	93.3	28.7	66.0	95.1	0.89	3.08	0.72	1.14	0.72	0.41
25	75	25	24.4	18.0	35.7	89.5	93.4	94.0	12.2	94.5	93.1	1.18	1.75	0.43	2.05	0.62	0.49
25	75	50	9.8	67.1	78.3	69.7	93.1	93.5	3.5	93.3	94.0	1.34	2.02	0.49	1.92	0.64	0.48
25	75	75	3.3	97.6	96.9	19.9	92.5	89.7	1.7	84.0	96.3	1.59	3.22	0.74	1.58	0.73	0.47
50	25	25	91.8	95.8	95.1	76.6	94.8	94.0	71.1	94.3	76.8	0.70	1.01	0.30	1.67	0.65	0.43
50	25	50	91.4	43.8	55.6	84.9	95.1	93.0	72.1	81.4	78.7	0.71	1.54	0.42	1.57	0.70	0.44
50	25	75	92.0	2.1	8.6	98.6	95.4	95.8	76.1	58.5	84.3	0.70	3.15	0.74	1.44	0.88	0.47
50	50	25	72.5	76.5	83.0	90.7	93.4	95.3	34.0	94.9	79.0	0.84	1.23	0.35	1.86	0.67	0.47
50	50	50	46.1	96.0	95.8	96.4	95.1	95.1	14.4	89.9	81.9	0.98	1.65	0.44	1.74	0.71	0.47
50	50	75	45.2	17.2	33.6	90.8	95.3	95.6	17.1	64.4	87.6	0.98	3.17	0.75	1.48	0.85	0.48
50	75	25	19.0	13.2	30.2	95.9	94.3	94.3	8.7	95.0	83.8	1.22	1.75	0.43	2.52	0.73	0.58
50	75	50	3.9	65.5	77.9	93.0	94.3	93.8	1.5	93.5	86.8	1.47	2.05	0.50	2.37	0.75	0.57
50	75	75	0.8	97.4	97.0	55.3	93.7	92.8	0.3	83.0	92.4	1.81	3.27	0.74	1.87	0.84	0.55
75	25	25	92.1	95.8	95.5	20.1	91.5	89.9	73.9	94.5	42.6	0.70	1.01	0.30	3.30	1.03	0.69
75	25	50	91.8	41.6	55.8	19.6	93.1	87.9	74.8	80.3	46.6	0.70	1.55	0.43	3.16	1.06	0.68
75	25	75	92.4	1.5	6.6	50.6	95.3	90.3	80.1	55.4	53.9	0.69	3.30	0.77	2.86	1.22	0.68
75	50	25	73.7	74.5	81.4	36.7	89.9	92.8	36.2	95.0	46.0	0.84	1.24	0.35	3.40	1.04	0.71
75	50	50	43.5	95.6	96.1	45.0	92.7	93.0	14.0	88.8	51.8	0.99	1.68	0.44	3.28	1.05	0.69
75	50	75	42.9	11.7	26.5	74.0	95.3	92.0	16.8	60.2	57.7	1.01	3.38	0.78	2.86	1.21	0.69
75	75	25	17.5	10.2	24.8	78.9	89.2	94.3	9.0	94.9	54.2	1.25	1.77	0.44	3.99	1.06	0.81
75	75	50	2.10	55.0	72.9	86.2	91.9	94.9	0.5	94.0	59.6	1.57	2.07	0.50	3.85	1.07	0.80
75	75	75	0.1	96.5	96.2	97.4	94.5	95.3	0.0	80.8	68.6	2.13	3.37	0.77	3.13	1.21	0.77

Table 12: Coverage probabilities and median 95% CI width for different degrees of censoring in $X_{(1)}$,$X_{(2)}$, and Y when the correlation between $X_{(1)}$ and $X_{(2)}$ is 0.75

6 Preliminary analysis: *Deepwater Horizon* oil spill response and clean-up efforts

6.1 Model comparison 2

In our second modeling comparison, we were interested in how well each model would fit the hexane data (accounting for complexity). Our first model was the multivariate framework where we allowed hexane to be dependent on both xylene and toluene exposures (called 3-variable model). We considered these two covariates because the non-censored measurements of toluene and xylene were not correlated significantly (Table 13). This allowed us to avoid multicollinearity concerns. Second, we consider two bivariate models for hexane with only xylene as a predictor and only toluene as a predictor. Finally, our last model considered modeling hexane alone (not dependent on THC, xylene, or toluene) using an ANOVA model. We assessed this model comparison using WAIC_Y. Variability in WAIC_Y was also assessed by changing model seeds and doing 100 runs for each model type.

Results of our second model comparison are shown in Table 14. In this model comparison, WAIC_Y was lowest for the 3-variable model like in comparison 1; therefore, this model was preferred. Again, the magnitude of the difference in WAIC_Y values was small, but a test of variability indicated that this difference was significant at the alpha level of 0.10. The ANOVA model of hexane had the highest WAIC_Y indicating that modeling hexane dependent on xylene or toluene was useful. $LPPD$ and P_2 estimates followed similar patterns to model comparison 1.

Chemical	Count Non-censored	Percent Censored	Non-censored Correlations Hexane	Toluene	Xylene
Hexane (Y)	53	12		0.42 (N=40)	0.25 (N=44)
Toluene	45	25			0.01 (N=38)
Xylene	48	20			

Table 13: Characteristics of *Ocean Intervention III* observations (N=60) from July 16 to September 30. We report count, percent censoring, and non-censored correlations between hexane, toluene, and xylene.

Model	Predictors	$LPPD$ Estimate	P_2 Estimate	WAIC$_Y$ Estimate	(5%, 95%)
3-Variable	Toluene, Xylene	-81.6	4.6	172.4	(172.3, 172.5)
Bivariate 1	Toluene	-83.5	3.3	173.6	(173.5, 173.7)
Bivariate 2	Xylene	-86.4	3.4	179.6	(179.5, 179.7)
ANOVA		-89.4	1.7	182.1	(182.1, 182.1)

Table 14: Model comparison for the *Ocean Intervention III* Data. Comparing WAIC$_Y$ values for the 3-variable model, bivariate models, and ANOVA model for hexane.WAIC$_Y$ estimates are reported for the same model seed of the program. The 5th and 95th percentiles of 100 runs with different model seeds are reported for WAIC$_Y$.