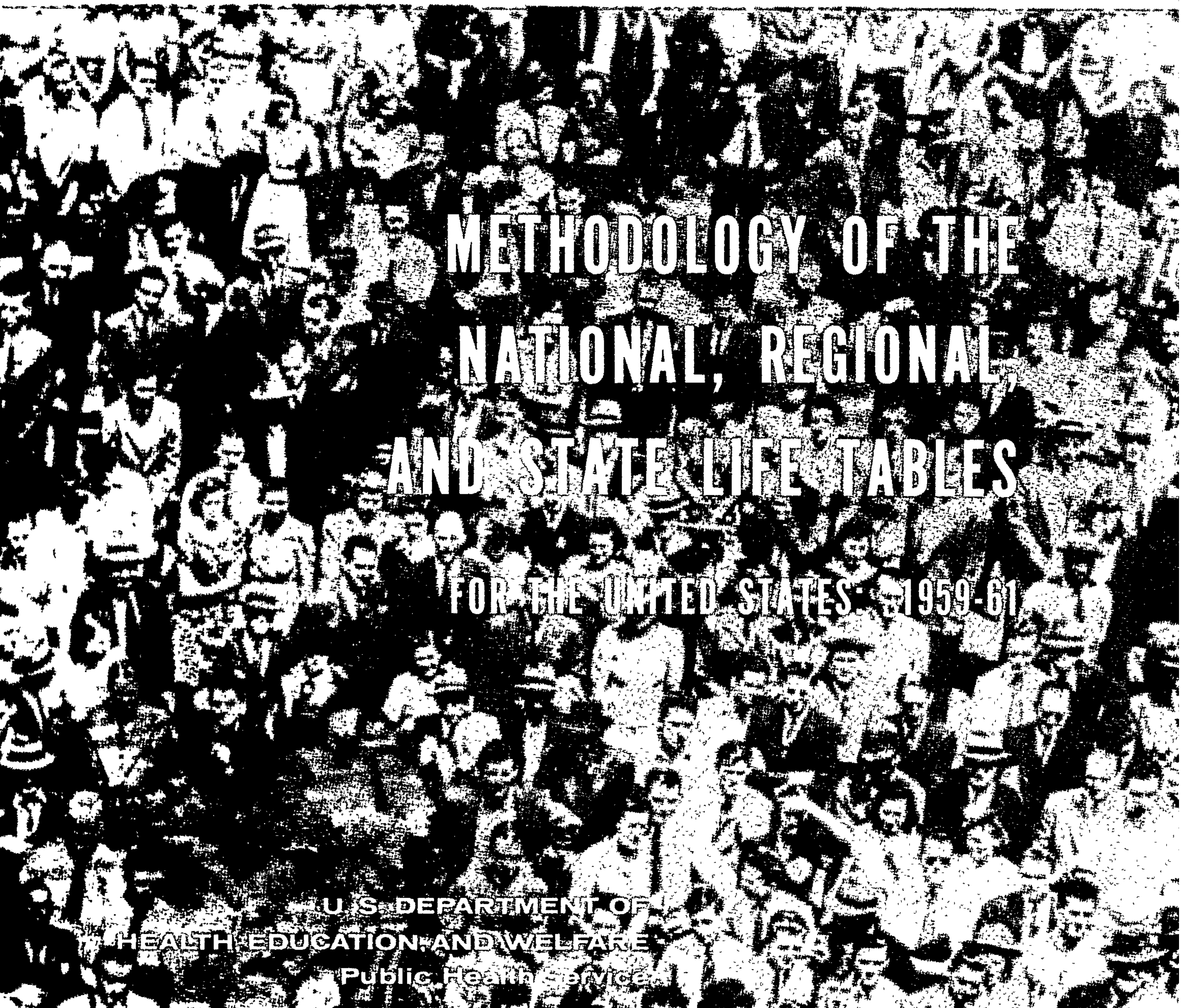
LIFE TABLES: 1959-61
VOLUME 1 - NO. 4



METHODOLOGY OF THE NATIONAL, REGIONAL, AND STATE LIFE TABLES

FOR THE UNITED STATES: 1959-61

U. S. DEPARTMENT OF
HEALTH, EDUCATION, AND WELFARE
Public Health Service

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U.S. DEPARTMENT OF HEALTH, EDUCATION, AND WELFARE
John W. Gardner, Secretary

PUBLIC HEALTH SERVICE
William H. Stewart, Surgeon General

Washington, D.C. October 1967

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FOREWORD

The preparation of the decennial life tables, 1959-61, was the joint effort of the Office of the Actuary of the Social Security Administration and the National Center for Health Statistics. The Office of the Actuary was responsible for developing the methods for constructing the life tables. The National Center for Health Statistics was responsible for directing and coordinating the project and for programming and calculating the life tables on its computer and for publishing the resulting reports.

Mr. Zenas M. Sykes and Mr. Francisco Bayó of the Social Security Administration developed the methods of constructing the life tables. Mrs. Florence K. Koons and Mr. David G. Halmstad of the National Center for Health Statistics prepared the computer programs. Dr. T. N. E. Greville is a mathematical consultant to the National Center for Health Statistics. He prepared this report and worked closely with Mr. Bayó in the development of the methods for constructing these life tables. Dr. Donald R. Schuette prepared the mathematical appendix to the report.

Monroe G. Sirken, Ph.D., Chief
Division of Health Records Statistics

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METHODOLOGY OF THE
NATIONAL, REGIONAL, AND STATE LIFE TABLES
FOR THE UNITED STATES: 1959-61

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INTRODUCTION

This report describes the methodology employed in the preparation of the decennial life tables for 1959-61 for the United States, individual States, the nine geographic divisions established by the Bureau of the Census, and metropolitan and nonmetropolitan areas as defined in the Census publications.[1] The methodology involved in the development of 1959-61 life tables for the United States by causes of death is highly specialized, and is given in the report containing these tables.[2] It will be assumed that the reader is acquainted with the definitions of the usual life table functions as given in other reports of this series. The calculations to be described later were made on the IBM 1401 and 1410 computers of the National Center for Health Statistics. Methodological studies in connection with the 1959-61 decennial life tables were initiated by Zenas M. Sykes, former Assistant Chief Actuary, Social Security Administration. The basic methodology followed in obtaining life table mortality rates for ages under 5, the idea of obtaining 5-year survival rates for ages 5 and above by equating the 5-year central death rates of the life table to those of the actual population, and the idea of using an iterative approximation procedure in this connection originated with him. The use of the separation factors ${}_5f_x$ to secure convergence of the iterative approximation procedure was the suggestion of Francisco Bayó, Deputy Chief Actuary, Social Security Administration. Methodological decisions necessitated by special problems arising in connection with the determination of life table mortality rates at ages 5 and above, and those peculiar to the life tables for States and other geographic subdivisions, were made primarily by him (sometimes in consultation with Monroe G. Sirken, Chief, Division of Health Records Statistics, National Center for Health Statistics, and with the writer of this report). The detailed examination described herein of the effects on life-table values of (1) the assumption of equality of 5-year central death rates for the actual population and the hypothetical life-table population, and (2) the use of April 1, 1960, populations without adjustment as if they were July 1 populations is also due to Mr. Bayó.

[1] The last named application required additional assumptions in order to provide for gaps in the tabulations of the underlying birth data and population data at ages 85 and over. For a description of these special adjustments, see National Center for Health Statistics, "Life Tables for Metropolitan and Nonmetropolitan Areas," *Life Tables, 1959-61*. To be published.

[2] National Center for Health Statistics: United States life tables by causes of death, 1959-61. *Life Tables, 1959-61*. To be published.

Table 1. Enumerated population and age-specific death rates of the nonwhite population by sex, ages 40-79: United States, 1959-61

Age x	Male: Enumerated population at ages x to x + 4, April 1, 1960	Male: ${}_5m_x^{pop}$ [a]	Male: ${}_5m_x^{pop} / {}_5m_{x-5}^{pop}$	Female: Enumerated population at ages x to x + 4, April 1, 1960	Female: ${}_5m_x^{pop}$ [a]	Female: ${}_5m_x^{pop} / {}_5m_{x-5}^{pop}$
40	558,843	0.0087		618,380	0.0065	
45	529,746	0.0121	1.39	564,577	0.0090	1.38
50	448,806	0.0186	1.54	463,620	0.0140	1.56
55	398,646	0.0248	1.33	405,008	0.0192	1.37
60	287,655	0.0390	1.57	304,124	0.0295	1.54
65	246,956	0.0502	1.29	271,730	0.0340	1.15
70	166,866	0.0630	1.25	181,024	0.0467	1.37
75	104,143	0.0741	1.18	114,098	0.0576	1.23

[a]Reported deaths at ages x to x + 4, 1959-61, divided by 3 times the enumerated population.

PRELIMINARY ADJUSTMENT OF DATA

In the preparation of the 1959-61 decennial life tables no allowance was made for possible incompleteness in the enumeration of the population or in registration of births or deaths. It is true, however, that the well known tendency toward underenumeration of infants in the census was recognized to the extent that for ages under 2, birth statistics rather than population data were used as an ingredient in the calculation of mortality rates. With this qualification, the official statistics produced by the Bureau of the Census and the National Center for Health Statistics were used without modification or adjustment to the extent that it was considered practicable to do so. In fact, the only important adjustment made in the underlying data involved a redistribution of the nonwhite population affecting the age groups 55-59 and 60-64.

The need for this adjustment arises from a preference for 1900 as the year of birth among nonwhites enumerated in the 1960 census. In this census the question regarding age asked for the date of birth (instead of the age in completed years, as in previous censuses). The age in completed years on the census date, April 1, 1960, was then calculated from the date of birth given. If a respondent reported a date of birth before April 1 in the year 1900, his age would be calculated as 60; for a 1900 birth date after April 1, the age 59 would be assigned. Thus, if there were a group of persons whose year of birth was erroneously reported as 1900 and if the reported dates of birth were more or less uniformly distributed over the months of the year, about one-fourth of the group would be assigned to age 60 and about three-fourths to age 59. If the erroneous reporting of year of birth involved understatement and overstatement of age about equally often, the net result would be an overstatement of the population at ages 55-59 at the expense of the population at ages 60-64.

Evidence that this has occurred is provided by table 1. In the columns showing the enumerated populations on April 1, 1960, it can be observed that the decrease from the age group 55-59 to the age group 60-64 is unexpectedly large in comparison with the corresponding population differences for other pairs of consecutive 5-year age groups. Moreover, the ratio of the death rate for the age group 60-64 to that for the age group 55-59 is unexpectedly large in both cases. The use of these death rates without adjustment in the construction of life tables would produce spurious points of inflection in the curve of rates of mortality.

The redistribution was made by fitting a second-degree polynomial to the populations in the four consecutive 5-year age groups 50-54, 55-59, 60-64, and 65-69 in such a way that the populations in the age intervals 50-54, 55-64, and 65-69 are exactly reproduced (but not 55-59 and 60-64).

If u_x denotes the reported population at ages x to $x+4$ and v_x the corresponding adjusted population, the adjusted populations v_{55} and v_{60} are given by the following formulas:

$$v_{55} = \frac{1}{6}(u_{50} + 3u_{55} + 3u_{60} - u_{65})$$

$$v_{60} = \frac{1}{6}(-u_{50} + 3u_{55} + 3u_{60} + u_{65}) \cdot$$

This redistribution was carried out for the nonwhite population of the United States and all geographic subdivisions, including metropolitan and nonmetropolitan areas.

A less significant adjustment results from the fact that census tabulations by age based on the full count of the population of the United States and its subdivisions ended with a final age group 85 and over. Subdivision of this grouping into 5-year age groups up to age 100, and a final group 100 and over, was available only on the basis of a 25-percent sample of the population. As the more detailed age classification was needed for life-table purposes, and as the populations aged 85 and over based on the (inflated) sample differed from those based on the complete count, an adjustment was necessary. Accordingly, an adjustment factor was computed for each population category (by sex, color, and geographic area) for which a life table was to be constructed. This adjustment factor was obtained by dividing the population aged 85 and over in the given category based on the complete count by the corresponding population based on the inflated sample. Then, for the given category, the figure derived from the sample for each detailed age grouping over age 84 was multiplied by the adjustment factor. As a result, the adjusted figures for the various age groupings over age 84 then added up to the population aged 85 and over for the given category, as obtained from the complete count.

A further relatively minor adjustment relates to the fact that the tabulations of deaths include a relatively small number for which the age is not reported. The assumption was made that these deaths were distributed among the various age groups in the same proportions as the deaths for which the age was reported. In order to give effect to this assumption, an adjustment factor was again computed for each population category for which a life table was to be constructed. In this case, the adjustment factor was obtained by dividing the total number of deaths reported for the given category for the 3-year period 1959-61 by the total less the number for which the age was not reported. The number of deaths reported in each age group for the given category was then multiplied by the adjustment factor.

CALCULATION OF "BASIC" LIFE-TABLE VALUES

General

The underlying data used in the preparation of each of the 1959-61 decennial life tables consisted of (1) reported deaths occurring in the 3-year period classified by age at death, (2) enumerated populations classified by age on the census date, April 1, 1960, and (3) total registered births for each of the calendar years 1957 to 1961, inclusive. Populations and deaths were available by single years of age under age 5, then by 5-year age groups up to age 99, with a final age group 100 and over. In each case the age referred to is the age in completed years: that is, the exact age on the individual's last birthday. In addition, deaths occurring at ages under 1 year were available for four subdivisions of this first year of life: 0-1 day, 1-3 days, 3-28 days, and 28-365 days. Life-table values were calculated for these subdivisions of the first year (but not published in the case of the State life tables), and by single years of age throughout the remainder of the life span.

In calculating the life-table values, fundamentally different procedures were followed for (1) ages under 2, (2) ages 2-4, and (3) ages 5 and over. In each of these three cases, however, the procedure may be regarded as consisting of three phases: (i) calculation from the underlying data of some "basic" life-table value or quantity for each age or age interval involved; (ii) calculation from the "basic" values of a column of values of ℓ_x, the number surviving to age x in the life table cohort starting with ℓ_0=100,000 live births, for all integer values of x and for those fractional values which are terminal ages of the subdivisions of the first year of life for which deaths were available; and (iii) calculation from the column of ℓ_x values of all the life-table values to be published, appropriately rounded, and edited so that the most obvious arithmetic relationships among the columns of the life table will hold exactly for the published figures. In each case, phase (i) is the decisive, and relatively troublesome phase, while phases (ii) and (iii) are relatively routine and straightforward. Though this section of the report deals primarily with phase (i), it will be necessary in a few instances to anticipate some features of phase (ii) in order to provide a sufficiently clear explanation of certain procedures involved in phase (i).

Ages Under 2

For these ages the "basic" life-table value is ${}_td_x$, the number of deaths occurring between exact ages x and $x+t$ in the life-table cohort commencing with ℓ_0 live births. This was calculated by the formula

$$ (1) \qquad {}_td_x = \ell_0 \, {}_tD_x / {}_tE_x, $$

where ${}_tD_x$ denotes the number of deaths (adjusted as described earlier for nonreporting of age) occurring in the 3-year period 1959-61 between exact ages x and $x+t$, and ${}_tE_x$ denotes the appropriate denominator as indicated in table 2. These denominators are based on the assumption of uniform distribution over the calendar year of the births of 1957, 1958, 1960, and 1961. In each case ℓ_0 is taken as 100,000.

Ages 2-4

For these ages the "basic" life-table value is q_x, the fraction or proportion of a group of persons at exact age x who are expected to die before attaining age $x+1$. If m_x denotes the ratio d_x/L_x, commonly called the central death rate, then it is well known[3] that on the assumption of uniform distribution of deaths over the year of age

$$ (2) \qquad q_x = \frac{2m_x}{2+m_x} $$

This formula was used to obtain q_2, q_3, and q_4, m_x being calculated in each case from the underlying data, as will now be explained.

If D_x denotes the adjusted number of deaths in a population category at age x (in completed years) occurring in 1959-61 and P_x denotes the population at age x in the middle of the period, then[3]

$$ (3) \qquad m_x = \frac{D_x}{3P_x}, $$

at least approximately. The middle of the 3-year period would be July 1, 1960; however, it was decided that no significant error would result from using in this connection populations enumerated in the 1960 census (as of April 1, 1960) without any adjustment for possible change in population during the 3-month period. A more detailed discussion of this assumption appears later in this report.

On the other hand, since the deaths occurring in a given single year of age during 1959-61 were drawn from three consecutive annual cohorts of the population, it was considered that the accuracy of the calculation of these m_x values would be improved by replacing $3P_x$ in the denominator of (3) by the sum of the populations at age $x-1$, x, and $x+1$. Thus, the formula becomes

$$ (4) \qquad m_x = \frac{D_x}{P_{x-1}+P_x+P_{x+1}}. $$

This formula, in combination with (2), was used at ages 2 and 3.

[3]Spiegelman, M.: *Introduction to Demography.* Chicago. The Society of Actuaries, 1955. p. 78.

Table 2. Denominators ${}_tE_x$ used in calculating ${}_td_x$ for ages under 2

Age interval x to x + t	Denominator of ${}_td_x$
0-1 day	$\frac{1}{730}(B_{1958} + 730\ B_{1959} + 730\ B_{1960} + 729\ B_{1961})$
1-3 days	$\frac{1}{730}(4B_{1958} + 730\ B_{1959} + 730\ B_{1960} + 726\ B_{1961})$
3-28 days	$\frac{1}{730}(31\ B_{1958} + 730\ B_{1959} + 730\ B_{1960} + 699\ B_{1961})$
28-365 days	$\frac{1}{730}(393\ B_{1958} + 730\ B_{1959} + 730\ B_{1960} + 337\ B_{1961})$
1-2 years	$\frac{1}{2}(B_{1957} + 2\ B_{1958} + 2\ B_{1959} + B_{1960})$

NOTE: B_z denotes the reported number of births occurring during the calendar year z for the population category (by sex, color, and geographic area) involved.

Consideration was given to the use of the same procedure at age 4 as well. However, this was not done because populations at age 5 were available only for the United States (not for geographic subdivisions), and it was desired to maintain a consistent methodology for both national and subnational life tables. Thus, formula (3), rather than (4), was used at age 4.

The procedure involving formula (4) was not used at age 1 because this would have involved the inclusion of P_0 in the formula, and this was considered unsuitable because of the well known tendency toward underenumeration of infants[4] and the relatively high mortality in the first year of life.

Ages 5 and Over

For ages 5 and over the "basic" life-table value was taken as ${}_5q_x = {}_5d_x/\ell_x$, the fraction or proportion of a group of persons at exact age x who are expected to die before attaining age $x+5$, x being a multiple of 5. In view of the availability of data only by 5-year age groups, these quantities could not be calculated easily, if at all, directly from the underlying data. However, the latter do yield at once the 5-year central death rate for the population

$$ {}_5m_x^{pop} = \frac{{}_5D_x}{3\,{}_5P_x}, \tag{5}$$

where ${}_5P_x$ denotes the enumerated population at ages x to $x+4$, inclusive. Similarly, a 5-year central death rate for the life table is defined by

$$ {}_5m_x = \frac{{}_5d_x}{{}_5L_x} = \frac{\ell_x - \ell_{x+5}}{T_x - T_{x+5}}. \tag{6}$$

It was considered reasonable to determine the life-table values so that these two 5-year central death rates are equal for each 5-year age group, starting with 5-9 and ending with 90-94, in other words,

$$ {}_5m_x = {}_5m_x^{pop}. \tag{7}$$

A detailed analysis of this assumption is given in the next subsection of this report. We shall now discuss the problem of determining numerical values of ${}_5q_5, {}_5q_{10}, \ldots, {}_5q_{90}$ such that equation (7) holds for $x = 5, 10, \ldots, 90$.

The interpolation and other procedures used to calculate the remaining life-table values from given values of ${}_5q_5, {}_5q_{10}, \ldots$, will be described in detail later in connection with phases (ii) and (iii) of the process of life-table construction. For the present it will suffice to say that, in consequence of the procedures mentioned, the re-

[4]Ibid., pp. 35-36.

quired values of ${}_5L_x$ are given in terms of $\ell_4, \ell_5, \ell_{10}, \ldots, \ell_{105}$ by the following formulas:

(8) $${}_5L_5 = -1.808303\,\ell_4 + 4.446995\,\ell_5 + 2.623337\,\ell_{10} - .300185\,\ell_{15} + .037251\,\ell_{20} + .000905\,\ell_{25},$$

(9) $${}_5L_{10} = .449328\,\ell_4 - .790687\,\ell_5 + 2.848458\,\ell_{10} + 2.779262\,\ell_{15} - .328328\,\ell_{20} + .041967\,\ell_{25}$$

(10) $${}_5L_x = .0368\,\ell_{x-10} - .3104\,\ell_{x-5} + 2.7736\,\ell_x + 2.7736\,\ell_{x+5} - .3104\,\ell_{x+10} + .0368\,\ell_{x+15}$$

(for $x = 15, 20, \ldots, 90$). As explained later, mortality rates at ages 95 and over in all the life tables were based on Union Civil War veterans' mortality experience. Consequently, the values of ${}_5q_{95}$ and ${}_5q_{100}$ can be regarded as given, and we have, of course, the relations

(11) $$\ell_{100} = \ell_{95}(1 - {}_5q_{95})$$

(12) $$\ell_{105} = \ell_{100}(1 - {}_5q_{100}).$$

It is easily verified (and will be shown later) that both ℓ_4 and ℓ_5 are uniquely determined by the "basic" life-table values for ages under 5, and can therefore be regarded as known quantities. From (6) and (7) we obtain

(13) $${}_5m_x^{\text{pop}}\;{}_5L_x = {}_5d_x$$

(for $x = 5, 10, \ldots, 90$). In view of formulas (8), (9), and (10), and since ${}_5d_x = \ell_x - \ell_{x+5}$ and the values of ${}_5m_x^{\text{pop}}$ are given numerically by (5), equations (11), (12), and (13) constitute a system of 20 simultaneous linear equations in the 20 unknown quantities $\ell_{10}, \ell_{15}, \ldots, \ell_{105}$.

Such a system of simultaneous linear equations is called "nonsingular" if there is one and only one set of values of the unknown quantities satisfying the equations.[5] It is proved rigorously in a mathematical appendix to this report that the system of equations encountered here is actually nonsingular for a large class of situations likely to arise in the construction of a life table. However, even if the conditions given in the appendix, which guarantee nonsingularity, are not satisfied, it would by no means follow that the system is singular. A singular system of equations is, as the name implies, exceptional, and it is most unlikely that mortality rates for any actual population would give rise to such a system.

It would have been possible to regard the quantities $\ell_{10}, \ell_{15}, \ldots, \ell_{105}$ as the "basic" life-table values and to obtain them numerically by solving the system of linear equations described in the preceding paragraphs by one of the standard methods available for solving such systems. In retrospect, I am now of the opinion that this would have been the most efficient method of procedure.[6] However, at the time the work was actually done, it was not appreciated, either by me or by the others more immediately concerned with developing the methodology employed, that the problem could be reduced to the solution of a system of linear equations.

It was sought, therefore, to devise an iterative procedure[7] that would provide successively closer approximations to the values of ${}_5q_5, {}_5q_{10}, \ldots, {}_5q_{90}$. Experiments were made with several iterative procedures before one was found that actually converged to a definite set of limiting values of ${}_5q_x$. The procedure eventually adopted gives at the third iteration life-table values that substantially satisfy (7) at all ages. This procedure is suggested by an attempt to generalize formula (2), in a modified form, to the case of a 5-year age interval. The assumption

[5]For those readers who are acquainted with determinants it may be remarked that a system of n linear equations in n unknown quantities is nonsingular if and only if the determinant of the coefficients of the unknown quantities is not equal to zero.

[6]It would be especially advantageous if the calculations were to be made on an electronic computer for which "library" programs were already available for solving systems of simultaneous linear equations.

[7]After the 1959-61 life tables had been calculated, but before this report on methodology was completed, Nathan Keyfitz described a somewhat similar method of life-table construction, in "A Life Table That Agrees With the Data," *Journal of the American Statistical Association,* Vol. 61, No. 314, Part 1, pp. 305-312, June 1966. He does not assume that the central death rates of the life table and of the actual population are equal, but rather that they are connected by means of an assumed rate of population increase. On his assumptions the determination of the life-table functions does not reduce to a system of linear equations, and he suggests an iterative procedure similar in some respects to the one described here.

of uniform distribution of deaths over the 5-year interval would give

(14) $$ {}_5q_x = \frac{5\ {}_5m_x}{1+2.5\,{}_5m_x}. $$

While the uniform-distribution assumption is clearly not admissible over an interval of this length, the values given by (14) may nevertheless be good enough for "starting" values to begin an iterative procedure.

Moreover, it is evident that there must exist a quantity ${}_5f_x$ (in general not equal to 2.5) such that

(15) $$ {}_5q_x = \frac{5\ {}_5m_x}{1+{}_5f_x\ {}_5m_x} $$

Solving (15) for ${}_5f_x$ gives

$$ {}_5f_x = \frac{5}{{}_5q_x} - \frac{1}{{}_5m_x} = \frac{5\ell_x - {}_5L_x}{{}_5d_x}. $$

From the second expression for ${}_5f_x$ it is easily deduced that $5-{}_5f_x$ is the average number of years lived beyond age x by the ${}_5d_x$ members of the life-table cohort who die between exact ages x and $x+5$. From this last observation it is clear that the numerical value of ${}_5f_x$ is necessarily between 0 and 5.

Denoting by a superscript r a life-table value obtained at the rth iteration, the iterative procedure used may be described as follows:

(16) $$ {}_5q_x^r = \frac{5\ {}_5m_x^{pop}}{1+{}_5f_x^r\ {}_5m_x^{pop}} $$

(17) $$ {}_5f_x^r = \begin{cases} 2.5 & \text{for } r=1 \\ \dfrac{5}{{}_5q_x^{r-1}} - \dfrac{1}{{}_5m_x^{r-1}} & \text{for } r>1 \end{cases} $$

In other words, values of ${}_5q_5^1, {}_5q_{10}^1, \ldots, {}_5q_{90}^1$ were first calculated by (16), taking ${}_5f_x^1 = 2.5$. From these ${}_5q_x^1$ values and those of ${}_5q_{95}$ and ${}_5q_{100}$ based on the Union Civil War veterans' experience, $\ell_{10}^1, \ell_{15}^1, \ldots, \ell_{105}^1$ were obtained. Then, ${}_5L_5^1, {}_5L_{10}^1, \ldots, {}_5L_{90}^1$ were calculated by (8), (9), and (10), and ${}_5m_5^1, {}_5m_{10}^1, \ldots, {}_5m_{90}^1$ were obtained from (6). By means of (17) ${}_5f_x^2$ was then computed for each 5-year age interval, and ${}_5q_x^2$ was obtained from (16). The entire process was then repeated.

Suppose that at some stage[8]

$$ {}_5q_x^r = {}_5q_x^{r-1} $$

for $x = 5, 10, \ldots, 90$. Then (16) gives

$$ \frac{5}{{}_5q_x^{r-1}} = \frac{1+{}_5f_x^r\ {}_5m_x^{pop}}{{}_5m_x^{pop}} = \frac{1}{{}_5m_x^{pop}} + {}_5f_x^r $$

Substituting this result in (17) and simplifying gives

$$ {}_5m_x^{r-1} = {}_5m_x^{pop} $$

Thus, if the iterative procedure described converges to definite values of ${}_5q_5, {}_5q_{10}, \ldots, {}_5q_{90}$, these values must necessarily be the ones belonging to the life table whose ${}_5m_x$ values satisfy (7).

In the mathematical appendix it is proved rigorously that this process does converge to definite values if certain conditions are satisfied. It is likely that it would converge in many situations in which the conditions stated in the appendix are not fulfilled. As a practical matter, in every case that has been tested it was found that the ${}_5m_x$ values of the life table and the actual population were substantially equal after three iterations. To provide additional assurance, five iterations were performed for each life table. In other words, the fifth set of values ${}_5q_x$ to be obtained (counting the initial approximations obtained by taking ${}_5f_x^1=2.5$) were the ones used in subsequent calculations.

[8]Strictly speaking, convergence of the iterative process does not imply that the ${}_5q_x$ values at two successive stages will ever be exactly equal, but rather that, for r sufficiently large, all the differences ${}_5q_x^r - {}_5q_x^{r-1}$ can be made smaller (in absolute value) than any preassigned positive quantity ϵ, however small. If this is indeed the case, it is easily shown, by a suitable modification of the argument, that it then follows that, for sufficiently large r, all the differences ${}_5m_x^r - {}_5m_x^{pop}$ can be made smaller in absolute value than any preassigned positive quantity η. This implies the existence of a unique set of limiting values of ${}_5q_x$ such that (7) holds exactly.

${}_5m_x^{pop}$ as a Goal for ${}_5m_x$

Assumption (7), which equates the 5-year central death rates of the actual population and the hypothetical life-table population, may be thought to introduce a systematic bias. It may be argued that in general the 5-year central death rates of the actual population should be lower than those of the life-table population, because of the assumption of identical cohort sizes in the life table as compared with the increasing cohort sizes of practically all real populations. Within each 5-year age group, the assumption of equal cohorts gives too much weight to mortality at the older ages.

It may be pointed out, however, that the noncentrality of the census date (April 1, 1960, rather than July 1, 1960) would be expected to introduce a systematic bias in the opposite direction. As the July 1 population would in general be larger than that on April 1, the use of the latter would tend to produce overstatement of the central death rate. Table 3 shows the results of some calculations based on the 1959-61 United States life table for the total population, designed to test the effects in this instance of increasing cohort size and noncentrality of the census date. Cohort size was assumed to change each year by a constant percentage. It was found that the resulting percentage differences in the ${}_5m_x$ values were of the order of about 15 percent to 20 percent (see the second column of table 3) of the annual percentage change in cohort size. This means, for example, that a 2 percent annual increase in cohort size would produce an understatement of ${}_5m_x$ of the order of 0.3 to 0.4 percent.

The third column of table 3 shows that these errors tend to be more than offset by those resulting from the noncentrality of the census date. It may be noted that, while the net percentage error in ${}_5m_x$ (expressed as a percentage of the assumed annual increase in cohort size) covers a fairly broad range, the large percentages occur at ages where there are few deaths, so that the net effect on the ℓ_x values of the life table is slight. In any event, for a 2 percent annual increase in cohort size the largest net relative error would be about 1 percent.

In the less usual case of decreasing cohort size in the actual population, each of the two biases mentioned would tend to act in the opposite direction to that described above. Thus their effects would again be offsetting, but opposite in sign.

In summary it is believed that the net errors resulting from the combination of assumption (7) and the use of census populations as if they were central are, for practical purposes, negligible, and that the use of the assumptions in question is fully justified.

Table 3. Estimate of relative error in ${}_5m_x$ due to increasing cohort size and noncentrality of census date (expressed as percentage of assumed annual percentage increase in cohort size)[a]

Age interval	Error due to increasing cohort size only	Net error due to increasing cohort size and April 1 census date
	Percentage	
5-9-----	20	50
10-14---	-24	-6
15-19---	-21	-1
20-24---	-5	19
25-29---	-4	21
30-34---	-12	10
35-39---	-17	4
40-44---	-19	2
45-49---	-19	1
50-54---	-17	4
55-59---	-17	4
60-64---	-16	5
65-69---	-16	5
70-74---	-17	4
75-79---	-18	2
80-84---	-18	2
85-89---	-20	1
90-94---	-15	7
95-99---	-9	14
100-104-	-8	15

[a] For example, if there were a 2 percent annual increase in cohort size, the relative error in the age interval 35-39 due to increasing cohort size only would be -17 percent of 2 percent or -0.34 percent.

NOTE: These are first approximations based on the 1959-61 U.S. life table for the total population.

CALCULATION OF ℓ_x VALUES FROM THE "BASIC" VALUES

Ages Under 5

At ages under 2 the "basic" life-table values were the values of ${}_td_x$ for successive age intervals, calculated on the assumption that $\ell_0 =$ 100,000. The value of ℓ_x for the terminal age of each age interval was obtained by successive application of the formula

$$\ell_{x+t} = \ell_x - {}_td_x.$$

At ages 2, 3, and 4, where the "basic" life-table values were the values of q_x, successive application of the formula

$$\ell_{x+1} = \ell_x(1-q_x) \tag{18}$$

yielded the values of ℓ_3, ℓ_4, and ℓ_5.

Interpolation Procedure at Ages 5 and Over

At ages 5 and over the "basic" values were those of ${}_5q_x$ by 5-year intervals from ${}_5q_5$ to ${}_5q_{90}$. Starting with ℓ_5, successive application of the formula

$$\ell_{x+5} = \ell_x(1-{}_5q_x) \tag{19}$$

gave the values of $\ell_{10}, \ell_{15}, \ldots, \ell_{95}$, and ℓ_{100} and ℓ_{105} were obtained from (19) by means of the values of ${}_5q_{95}$ and ${}_5q_{100}$ based on the Union Civil War veterans' experience. As ℓ_x values by single years of age were required, interpolation was performed on the quinquennial ℓ_x values using interpolation coefficients developed by H. S. Beers, which are shown in table 4.

In this application the coefficients given in the table for calculating u_1, u_2, and u_3 were not used, and those for u_4 were used only indirectly in connection with a special device to insure smooth junction of the values of ℓ_6, ℓ_7, etc., with those of ℓ_4 and ℓ_5 already obtained. Direct application of the table to calculate ℓ_4 would give

$$\ell_4 = .0819\,\ell_0 + 1.0689\,\ell_5 - .1666\,\ell_{10} - .0126\,\ell_{15} + .0399\,\ell_{20} - .0115\,\ell_{25}.$$

Solving this equation for ℓ_0 (which we now call ℓ_0^*) gives

$$\ell_0^* = \frac{1}{.0819}(\ell_4 - 1.0689\,\ell_5 + .1666\,\ell_{10} + .0126\,\ell_{15} - .0399\,\ell_{20} + .0115\,\ell_{25}).$$

A fictitious value ℓ_0^* calculated by this formula was used instead of the true ℓ_0 in calculating ℓ_6, ℓ_7, ℓ_8, ℓ_9, ℓ_{11}, ℓ_{12}, ℓ_{13}, and ℓ_{14} by means of table 4. It follows from the derivation of the formula for ℓ_0^* that the use of this value to calculate ℓ_4 by means of table 4 would reproduce the previously calculated value of this quantity. Moreover, the smoothness inherent in the Beers coefficients insures that ℓ_4 and ℓ_5 will form a smoth progression with ℓ_6 and succeeding values.

The use of the fictitious value ℓ_0^* is justified on the ground that the actual value of ℓ_0 cannot be regarded as forming a smooth progression with the values of ℓ_5, ℓ_{10}, etc., because of the relatively high mortality occurring in infancy.

In carrying out the iterative process already indicated, which is designed to secure equality of 5-year central death rates in the life table and in the actual population, the entire interpolation procedure just described (including the special adjustment to provide smooth junction at age 5) was used at each step of the iteration. As previously explained, the iterative process requires the calculation at each step of ${}_5L_x$ for each 5-year age interval between ages 5 and 95. At all ages 1 and above it was assumed that

$$L_x = \tfrac{1}{2}(\ell_x + \ell_{x+1}),$$

and, of course,

$${}_5L_x = L_x + L_{x+1} + L_{x+2} + L_{x+3} + L_{x+4}.$$

Table 4. Beers' interpolation coefficients for performing interpolation on quinquennial values of a function u_x to obtain values at intervals of unity ("minimized fifth-difference formula with smoother ends")[a]

Coefficients to be used in first two intervals

Coefficients of u_x to obtain:	x					
	0	5	10	15	20	25
u_1	.6667	.4969	-.1426	-.1006	.1079	-.0283
u_2	.4072	.8344	-.2336	-.0976	.1224	-.0328
u_3	.2148	1.0204	-.2456	-.0536	.0884	-.0244
u_4	.0819	1.0689	-.1666	-.0126	.0399	-.0115
u_6	-.0404	.8404	.2344	-.0216	-.0196	.0068
u_7	-.0497	.6229	.5014	-.0646	-.0181	.0081
u_8	-.0389	.3849	.7534	-.1006	-.0041	.0053
u_9	-.0191	.1659	.9354	-.0906	.0069	.0015

Coefficients to be used in subsequent intervals

Coefficients of u_x to obtain:	x					
	5m-10	5m-5	5m	5m+5	5m+10	5m+15
u_{5m+1}	.0117	-.0921	.9234	.1854	-.0311	.0027
u_{5m+2}	.0137	-.1101	.7194	.4454	-.0771	.0087
u_{5m+3}	.0087	-.0771	.4454	.7194	-.1101	.0137
u_{5m+4}	.0027	-.0311	.1854	.9234	-.0921	.0117

[a]American Institute of Actuaries: The Record, Vol. XXXIV, Part 1, p.59. 1945.

For convenience, these formulas were combined with the interpolation coefficients in table 4 to obtain formulas (8), (9), and (10), which express ${}_5L_x$ directly in terms of $\ell_4, \ell_5, \ell_{10}, \ell_{15}$, etc.

Ending the Life Tables

As the underlying data at ages 95 and over were scanty and unreliable, life-table values calculated for these ages from the underlying data exhibit anomalous behavior, and would seldom, if ever, be suitable for inclusion in the published life tables. Accordingly, q_x values for ages 95 and over from the Union Civil War veterans' experience[9] were used instead of the calculated values. As a result, the q_x values at these ages are the same in all the life tables.

[9]Myers, R. J., and Shudde, L. O.: Mortality experience of Union Civil War veterans. *Transactions of the Society of Actuaries*, Vol. VII, pp. 63-68. Chicago, Mar. 1955.

In order to provide a smooth transition from the values based on the relevant underlying data to those based on Union Civil War veterans' experience, a linear blending of the two sets of mortality rates was used at ages 85 to 94, inclusive. The blending formula was

$$q_x = \frac{1}{11}\left[(95-x)\, q_x^C + (x-84)\, q_x^V\right],$$

where q_x denotes the value finally adopted, q_x^C the value calculated, by the formula

$$q_x = 1 - \ell_{x+1} / \ell_x ,$$

from the ℓ_x values obtained at the final stage of the iteration, and q_x^V the value based on the Union Civil War veterans' experience. Using the blended values of q_x for ages 85-94, and the values based on Civil War veterans' experience for ages 95 and over, values of ℓ_x were recalculated by (18) for ages 86 and over. This sequence of ℓ_x values was extended to age 139.

CALCULATION OF THE REMAINING LIFE-TABLE VALUES

At the conclusion of phase (ii) of the overall process of life table calculation, there was available, for each life table to be constructed, a sequence of values of ℓ_x extending from age 0 to age 139. These included values for all integral ages and for the ages of 1 day, 3 days, and 28 days. These ℓ_x values were carried out to 14 decimal places.

Using all 14 decimal places, the values of T_x, working from age 139 back to age 1, were calculated by the formula

$$T_x = T_{x+1} + \frac{1}{2}(\ell_x + \ell_{x+1}) .$$

T_{139} was taken as zero. For the subdivisions of the first year of life, the corresponding formula

$$T_x = T_{x+t} + \frac{t}{2}(\ell_x + \ell_{x+t})$$

was used. For ages 0, 1 day, 3 days, and 28 days, respectively, t was taken as 1/365, 2/365, 25/365, and 337/365: For each age up to and including age 109, ${}_tq_x$ and $\mathring{e}_x$ were obtained by the formulas

$${}_tq_x = 1 - \ell_{x+t} / \ell_x ,$$

$$\mathring{e}_x = T_x / \ell_x .$$

In these calculations 20 significant digits were retained in the computer.

Each life table was then cut off at age 110 (that is, "109-110" is the last line shown), and the four columns of figures thus far obtained were rounded: ℓ_x and T_x to the nearest integer, ${}_tq_x$ to the nearest fifth decimal place, and $\mathring{e}_x$ to the nearest second decimal place.

The values of ${}_td_x$ and ${}_tL_x$ were obtained by differencing the rounded ℓ_x and T_x columns:

$${}_td_x = \ell_x - \ell_{x+t} ,$$

$${}_tL_x = T_x - T_{x+t} .$$

SPECIAL ADJUSTMENTS IN LIFE TABLES FOR GEOGRAPHIC DIVISIONS AND STATES

For each of the 50 States and the District of Columbia, life tables were calculated for nine classifications by color and sex: the total population, total whites, total nonwhites, total males, total females, white males, white females, nonwhite males, and nonwhite females. Life tables were calculated also for the same nine classifications for each of the nine geographic divisions of the United States employed by the Bureau of the Census, and further for the two groups of metropolitan counties and nonmetropolitan counties within each of the nine geographic divisions.

Of the nine tables calculated for each State only five were published: those for the total population, white males, white females, nonwhite males, and nonwhite females. In the case of the tables for the nine geographic divisions (including their metropolitan and nonmetropolitan subdivisions), only four tables were published (that for the total population being omitted). Publication of the life tables for nonwhite males and nonwhite females was also omitted for certain areas for which it was considered that the amount of data involved was too small to produce reliable results. The criterion adopted was that if, for any geographic subdivision, the number of reported deaths at all ages for the 3-year period 1959-61 for either nonwhite males or nonwhite females (or both) was less than 2,000, both life tables for nonwhites were omitted from publication. Following this criterion, life tables for nonwhites were not published for the States of Alaska, Arizona, Colorado, Connecticut, Delaware, Idaho, Iowa, Kansas, Maine, Massachusetts, Minnesota, Montana, Nebraska, Nevada, New Hampshire, New Mexico, North Dakota, Oregon, Rhode Island, South Dakota, Utah, Vermont, Washington, West Virginia, Wisconsin, and Wyoming, for nonmetropolitan counties of the New England division, and for metropolitan counties of the Mountain division. In only two instances (the States of Arizona and West Virginia) was the number of nonwhite deaths less than 2,000 for one sex (females in both cases) and not for the other. If the same criterion had been applied to the life tables for white persons, publication of the tables for white males and white females in Alaska and Hawaii would have been omitted.

In most of the life tables that were published for geographic divisions and States, special adjustments were made at certain ages to correct or mitigate anomalous behavior of the life-table values, that may be attributed to the small numbers involved. After each life table intended for publication had been calculated and printed out, the q_x values for individual years of age (including subdivisions of the first year of life) were examined, and certain tests of consistency applied. The position was taken that all the other life-table functions are completely determined by the q_x values, and no tests need to be applied to them.

It was considered that, for each life table, the q_x values should decrease from age 0 to about age 10 or 11 and then increase to the early twenties. They should increase again from about age 30 to the end of the table. Strict increase in mortality rates with increasing age was not required between ages 20 and 30, because a slight dip in the mortality curve in this age range (due to motor-vehicle-accident deaths) is a feature of many of the life tables. Abrupt age-to-age changes in q_x values (indicated by large second differences) were also examined.

For each set of four life tables (by color and sex) for a given geographic area, it was considered that the q_x value for females at each age should be less than the corresponding value for males of the same color, with the possible exception of ages 90-95. (At ages 96 and over, the q_x values were the same in all the life tables, being those of the Union Civil War veterans' mortality experience.) Moreover, in every such set of four life tables, the q_x value for the white population at each age should be less than the corresponding value for the nonwhite population of the same sex, up to about age 70. If the values for nonwhite persons do become lower at about age 70 or later, they should remain lower. In other words, corresponding mortality curves for white and nonwhite persons should not be permitted to cross and recross a number of times. The criterion based on comparison of corresponding mortality rates for white and nonwhite persons was not applied, however, to life tables for Hawaii, California, and the Pacific division, where the nonwhite population is composed predominantly of ethnic groups having mortality rates closely comparable to those of the white population.

In every instance in which an adjustment was thought necessary, it was effected by redistributing by age the deaths in two or more adjacent age groups, so that the total deaths remained unchanged. In using this type of adjustment, the intention was to change the local shape of the mortality curve, while preserving the overall mortality level. In most cases, the redistribution by age was made in proportion to the deaths of the corresponding age intervals for a "standard" life table. For the most part, the life table for the United States for the same classification by race and sex was taken as the "standard." In a few cases involving State life tables, the corresponding life table for the geographic division containing the given State was used instead. In general, an effort was made to extend the adjustment over the smallest possible number of ages consistent with achieving the results desired. In a few recalcitrant cases, redistribution on the basis of a "standard" life table failed to

Table 5. Number of State[a] life tables in which special adjustments were made, by color, sex, and selected age intervals: United States, 1959-61

Age interval (between exact ages)	Number of life tables in which special adjustments were made[b]			
	White		Nonwhite	
	Male	Female	Male	Female
0-2	14	9	4	6
2-5	36	38	22	22
5-20	13	11	16	13
20-50	8	10	1	3
50-95	4	3	13	12
95-105	1	0	0	0
Total numbers of life tables published	51	51	25	25

[a]Including the District of Columbia.

[b]In some instances the age interval involved in a single redistribution of deaths by age included parts of two or more of the age intervals shown in the table. Thus the sum of the entries in any column, in general, somewhat exceeds the total number of separate redistributions made.

remove the observed anomalies, and redistribution of deaths by age in a more arbitrary manner was resorted to.

When an appropriate redistribution by age of the deaths in certain age intervals had been decided on, it was necessary to make the corresponding changes in the life-table functions. When an age interval involved in the redistribution was under age 2, the adjusted number of deaths was substituted for the original ${}_tD_x$ in formula (1). If age 2, 3, or 4 was involved, an adjusted value of m_x was calculated by (3) or (4), as appropriate, and an adjusted q_x was then given by (2). If a 5-year age interval was involved, the original ${}_5m_x^{pop}$ was replaced by an adjusted value calculated by (5). The corresponding adjusted value of ${}_5q_x$ was then calculated by (15), using for ${}_5f_x$ the corresponding value obtained from the life table for the United States for the same classification by color and sex. After making all such corrections, stages (ii) and (iii) of the process of constructing the life table, as described earlier, were repeated with the corrected values.

Table 5, which relates only to the life tables for States (including the District of Columbia) gives some idea of the number of special adjustments made. Fairly numerous special adjustments were made also in the life tables for the nine geographic divisions and their metropolitan and nonmetropolitan subdivisions; these are not reflected in the table.

The life tables for the total population of a State are also not included because, for the sake of consistency, all redistributions of deaths in any of the four component parts were incorporated into the table for the total population. Thus, many special adjustments in the life tables for the total population of a State would not have been made if these tables had been considered by themselves.

More adjustments were required in the age interval 2-5 years than in any other. This is because deaths by single years of age were used at ages 2, 3, and 4 (in completed years), and these numbers are small and subject to severe statistical fluctuations. Only 12 out of 51 State life tables for white males, and only 11 out of 51 for white females, did not require special adjustments at some ages. All 25 of the published State life tables for nonwhite males, and all but one (New York) of those for nonwhite females, required some special adjustments.

Because of the method of adjustment used (redistribution of deaths by age over two or more adjacent age intervals), it was not possible to make a change in one of the age intervals by which deaths were tabulated without affecting neighboring ones. In some instances a single redistribution of deaths affected parts of two or more of the age classifications shown in table 5. Therefore, the total of any column in the table slightly exceeds the number of single redistributions made. In only one instance (white males in Hawaii) was it necessary to include deaths at ages above 95 in a redistribution.

MATHEMATICAL APPENDIX

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INTRODUCTION

As pointed out in the main text of this report, it is conceivable (though unlikely in practice) that a set of values of ${}_5m_x^{pop}$ might be such that the system of linear equations consisting of equations (11), (12), and (13) has not a unique solution, but either no solution or a multiplicity of solutions. It is also conceivable that in some cases the iterative procedure described by equations (16) and (17) might fail to converge (even though there is a unique solution). It would be desirable, therefore, to know that if certain conditions are satisfied by the observed values of ${}_5m_x^{pop}$ there will always be a unique solution, and that if certain further conditions are satisfied, the iterative procedure must converge to a definite set of limiting values.

Such conditions are derived in this appendix. It should be emphasized, however, that while the respective conditions to be described guarantee (1) uniqueness of the solution and (2) convergence of the iterative process, it is considered unlikely that the central death rates for any actual population would give rise to either nonuniqueness or nonconvergence, even if the conditions are not satisfied.

On the other hand, it should perhaps be pointed out that there could be a situation in which there was a unique solution, but for practical reasons this solution would have to be rejected or modified. For example, it is mathematically possible for the values of $\ell_{10}, \ell_{15}, \ldots, \ell_{105}$ obtained by solving the system of equations to be such that the use of Beers' interpolation coefficients would give $\ell_x < \ell_{x+1}$ at some age x.

CONDITIONS FOR UNIQUENESS OF THE SOLUTION

In this part of the appendix the system of 20 linear equations in the 20 unknowns, $\ell_{10}, \ell_{15}, \ldots, \ell_{105}$, defined by equations (11), (12), and (13) will be examined for conditions under which the system is nonsingular, i.e., conditions under which the set of values for the unknowns that satisfies the equations is unique.

Equation (13), ${}_5m_x^{pop}\,{}_5L_x = {}_5d_x$, when ${}_5L_x$ is replaced by its equivalent in terms of $\ell_4, \ell_5, \ldots, \ell_{105}$, and ${}_5d_x$ is replaced by $\ell_x - \ell_{x+5}$, becomes the equations

$$\text{(A1)}\quad {}_5m_5^{pop}\,[-1.808303\ell_4 + 4.446995\ell_5 + 2.623337\ell_{10} - .300185\ell_{15} + .037251\ell_{20} + .000905\ell_{25}] = \ell_5 - \ell_{10},$$

$$\text{(A2)}\quad {}_5m_{10}^{pop}\,[.449328\ell_4 - .790687\ell_5 + 2.848458\ell_{10} + 2.779262\ell_{15} - .328328\ell_{20} + .041967\ell_{25}] = \ell_{10} - \ell_{15},$$

and

$$\text{(A3)}\quad {}_5m_x^{pop}\,[.0368\ell_{x-10} - .3104\ell_{x-5} + 2.7736\ell_x + 2.7736\ell_{x+5} - .3104\ell_{x+10} + .0368\ell_{x+15}] = \ell_x - \ell_{x+5}$$

for $x = 15, 20, \ldots, 90$.

It will be convenient to divide[10] these equations by the appropriate value of ${}_5m_x^{pop}$ and then rearrange so that terms involving unknowns appear on the left side and all known quantities including ℓ_4 and ℓ_5 are on the right side. The resulting equations with $\alpha_x = 1/{}_5m_x^{pop}$ are as follows:

$$\text{(A4)}\quad (\alpha_5 + 2.623337)\ell_{10} - .300185\ell_{15} + .037251\ell_{20} + .000905\ell_{25} = (\alpha_5 - 4.446995)\ell_5 + 1.808303\ell_4,$$

$$\text{(A5)}\quad (-\alpha_{10} + 2.848458)\ell_{10} + (\alpha_{10} + 2.779262)\ell_{15} - .328328\ell_{20} + .041967\ell_{25} = .790687\ell_5 - .449328\ell_4,$$

$$\text{(A6)}\quad -.3104\ell_{10} + (2.7736 - \alpha_{15})\ell_{15} + (2.7736 + \alpha_{15})\ell_{20} - .3104\ell_{25} + .0368\ell_{30} = -.0368\ell_5,$$

and

$$\text{(A7)}\quad .0368\ell_{x-10} - .3104\ell_{x-5} + (2.7736 - \alpha_x)\ell_x + (2.7736 + \alpha_x)\ell_{x+5} - .3104\ell_{x+10} + .0368\ell_{x+15} = 0$$

for $x = 20, 25, \ldots, 90$.

Equations (11) and (12) may be rearranged respectively as

$$\text{(A8)}\quad -{}_5p_{95}\ell_{95} + \ell_{100} = 0$$

$$\text{(A9)}\quad -{}_5p_{100}\ell_{100} + \ell_{105} = 0.$$

Equations (A4) to (A9) inclusive comprise the system of 20 linear equations to be examined. As previously indicated, the values of ${}_5p_{95}$ and ${}_5p_{100}$ based upon Union Civil War veterans' mortality experience may be regarded as given.

[10]Here the assumption is made that there is no value of x for which ${}_5m_x^{pop} = 0$. The procedure described in the main part of this report could not be applied, without some modification, to a set of central death rates that were not all positive. For, in view of (13), ${}_5m_x^{pop} = 0$ implies ${}_5d_x = 0$ and therefore $\ell_{x+5} = \ell_x$. This, in turn, implies that $\ell_{x+1} = \ell_{x+2} = \ell_{x+3} = \ell_{x+4} = \ell_x$. It is not difficult to show, however, that Beers' interpolation coefficients could give such a result only if ℓ_x were a constant (independent of x), which would mean that no deaths occurred at any age.

In fact, the values which were actually used were ${}_5p_{95} = .120378690$ and ${}_5p_{100} = .062156908$. Hence the only nonnumerical quantities other than the unknowns which appear on the left side of the equations are the quantities $\alpha_x = 1/{}_5m_x^{pop}$, $x = 5, \ldots, 90$. The latter, of course, vary among the various tables, and their values will determine whether or not the set of solution values of the unknowns is unique. Hence, in what follows it will be natural to expect that the conditions for a unique solution to the system of equations will depend upon the values of α_x, $x = 5, 10, \ldots, 90$.

Now, as indicated earlier, a system of n linear equations in n unknowns is nonsingular if and only if the determinant of the coefficients of the unknowns does not equal zero.[11] Hence, one way of proving nonsingularity would be to compute the value of the 20 x 20 determinant of coefficients and obtain a nonzero value. But, because there is really a different system of equations and hence a different determinant for each of the various national and subnational tables, a more general approach would be preferred. Such an approach is available in the form of a theorem concerning homogeneous linear equations.

A system of n homogeneous linear equations in n unknowns $w_1, w_2, \ldots, w_n$ is a system of the form

$$\begin{aligned} a_{11}w_1 + a_{12}w_2 + \ldots + a_{1n}w_n &= 0 \\ a_{21}w_1 + a_{22}w_2 + \ldots + a_{2n}w_n &= 0 \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ a_{n1}w_1 + a_{n2}w_2 + \ldots + a_{nn}w_n &= 0. \end{aligned}$$

In other words, a homogeneous system is one in which all the constants on the right sides of the equations have the value zero. Clearly $w_1 = 0$, $w_2 = 0, \ldots, w_n = 0$ is an obvious solution to this system, the so-called trivial solution. The theorem in question is that a set of n homogeneous linear equations in n unknowns has a solution other than the trivial solution if, and only if, the determinant of coefficients vanishes.[12]

[11]The reader may consult as a reference on this point Browne, Edward T., *Introduction to the Theory of Determinants and Matrices*, The University of North Carolina Press, Chapel Hill, 1958 (p. 57).

[12]Ibid., Corollary 23.4, p. 61.

If it can be shown that the homogeneous system corresponding to equations (A4) to (A9) inclusive, i.e., the system obtained when the right hand sides of equations (A4), (A5), and (A6) are replaced by zero, has only the trivial solution, then from the above-mentioned theorem it may be concluded that the determinant of coefficients is nonzero. But since the determinant of coefficients is the same for the original system of equations as it is for the corresponding homogeneous system, the nonsingularity of the original system will then have been established.

Accordingly, let $z_{10}, z_{15}, \ldots, z_{105}$ be values of the unknowns satisfying the homogeneous system corresponding to equations (A4) to (A9) inclusive. Then

$$\text{(A10)}\quad (\alpha_5 + 2.623337)z_{10} - .300185z_{15} + .037251z_{20} + .000905z_{25} = 0$$

$$\text{(A11)}\quad (-\alpha_{10} + 2.848458)z_{10} + (\alpha_{10} + 2.779262)z_{15} - .328328z_{20} + .041967z_{25} = 0$$

$$\text{(A12)}\quad -.3104z_{10} + (2.7736 - \alpha_{15})z_{15} + (2.7736 + \alpha_{15})z_{20} - .3104z_{25} + .0368z_{30} = 0$$

$$\text{(A13)}\quad .0368z_{10} - .3104z_{15} + (2.7736 - \alpha_{20})z_{20} + (2.7736 + \alpha_{20})z_{25} - .3104z_{30} + .0368z_{35} = 0$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\text{(A14)}\quad .0368z_{80} - .3104z_{85} + (2.7736 - \alpha_{90})z_{90} + (2.7736 + \alpha_{90})z_{95} - .3104z_{100} + .0368z_{105} = 0$$

$$\text{(A15)}\quad -{}_5p_{95}z_{95} + z_{100} = 0$$

$$\text{(A16)}\quad -{}_5p_{100}z_{100} + z_{105} = 0.$$

Multiplying equation (A10) by z_{10}, equation (A11) by z_{15} and so on down to equation (A16), which is multiplied by z_{105}, and adding produce, after considerable rearrangement, a result of the form $S = 0$ where

$$\begin{aligned}
\text{(A17)}\quad S = {} & 1.2741365(z_{10} + z_{15})^2 + 1.222636(z_{15} + z_{20})^2 \\
& + 1.2316[(z_{20} + z_{25})^2 + (z_{25} + z_{30})^2 + \ldots (z_{90} + z_{95})^2] \\
& + .215389345(z_{95} - z_{100})^2 + .031078454(z_{100} - z_{105})^2 \\
& + .1365745(z_{10} - z_{20})^2 + .1342165(z_{15} - z_{25})^2 \\
& + .1368[(z_{20} - z_{30})^2 + (z_{25} - z_{35})^2 + \ldots (z_{85} - z_{95})^2] \\
& .0184[(z_{90} + z_{100})^2 + (z_{95} + z_{105})^2] \\
& + .0188525(z_{10} + z_{25})^2 + .0184[(z_{15} + z_{30})^2 \\
& + \ldots (z_{80} + z_{95})^2] \\
& + 1.1937735z_{10}^2 + .129873z_{15}^2 + .0275895z_{20}^2 \\
& + .002131z_{25}^2 \\
& + .0184z_{85}^2 + .1368z_{90}^2 + 1.153010655z_{95}^2 \\
& + .735132201z_{100}^2 + .950521546z_{105}^2 \\
& + Q
\end{aligned}$$

where

$$\begin{aligned}
\text{(A18)}\quad Q = {} & \tfrac{1}{2}\alpha_{10}(z_{10} - z_{15})^2 + \tfrac{1}{2}\alpha_{15}(z_{15} - z_{20})^2 + \ldots \\
& \tfrac{1}{2}\alpha_{90}(z_{90} - z_{95})^2 \\
& + z_{10}^2(\alpha_5 - \tfrac{1}{2}\alpha_{10}) + z_{15}^2(\tfrac{1}{2}\alpha_{10} - \tfrac{1}{2}\alpha_{15}) \\
& + \ldots z_{90}^2(\tfrac{1}{2}\alpha_{85} - \tfrac{1}{2}\alpha_{90}) + z_{95}^2(\tfrac{1}{2}\alpha_{90}).
\end{aligned}$$

It is perhaps well to repeat at this point that what is to be shown is that for S to have value zero, $z_{10}, z_{15}, \ldots, z_{105}$ all must equal zero. But, because S is a weighted sum of squares of a number of real quantities, S can have value zero only when all of the quantities to be squared are themselves zero, provided all of the coefficients (weights) are positive. Inasmuch as all the coefficients of terms in S which do not involve α_x, $x = 5, 10, \ldots, 90$, are positive, the desired result is obtained whenever the α_x are such that all of the coefficients of the terms in Q are positive.

It is perhaps obvious that one way for all the coefficients of Q to be positive is that the value of ${}_5m_x^{pop}$ increase with age. For if ${}_5m_x^{pop} < {}_5m_{x+5}^{pop}$ for all x, then $\alpha_x > \alpha_{x+5}$ and $\frac{1}{2}\alpha_x - \frac{1}{2}\alpha_{x+5} > 0$ for all x. For $x=5$ all that is actually required is that $\frac{1}{2}\,{}_5m_5^{pop} < {}_5m_{10}^{pop}$, because then $2\alpha_5 > \alpha_{10}$ and $\alpha_5 - \frac{1}{2}\alpha_{10} > 0$. Hence a first result is the following:

Theorem A1: The system of linear equations defined by equations (11), (12), and (13) has one and only one solution for the unknowns $\ell_{10}, \ell_{15}, \ldots, \ell_{105}$ if ${}_5m_x^{pop} < {}_5m_{x+5}^{pop}$ for $x = 10, 15, \ldots, 90$, and $\frac{1}{2}\,{}_5m_5^{pop} < {}_5m_{10}^{pop}$.

Unfortunately, the conditions of Theorem A1 are inadequate for many tables because of the fact that in a number of instances ${}_5m_x^{pop} > {}_5m_{x+5}^{pop}$. This is quite common for male tables at $x = 20$ because of the impact of deaths due to motor-vehicle accidents. Hence, conditions on α_x values which encompass more situations must be sought.

Suppose there is an age u such that ${}_5m_{u-5}^{pop} < {}_5m_u^{pop}$ but ${}_5m_u^{pop} > {}_5m_{u+5}^{pop}$. Then $\alpha_{u-5} > \alpha_u$ and the coefficient of z_u^2 in Q is positive, but $\alpha_u < \alpha_{u+5}$ and the coefficient of z_{u+5}^2 is negative. However, a way to circumvent this difficulty is to rearrange the term with the negative coefficient and two others of Q as follows:

$$\tfrac{1}{2}\alpha_u(z_u - z_{u+5})^2 + z_u^2\left(\tfrac{1}{2}\alpha_{u-5} - \tfrac{1}{2}\alpha_u\right) + z_{u+5}^2\left(\tfrac{1}{2}\alpha_u - \tfrac{1}{2}\alpha_{u+5}\right)$$

$$= -\alpha_u z_u z_{u+5} + \tfrac{1}{2}\alpha_{u-5} z_u^2 + \left(\alpha_u - \tfrac{1}{2}\alpha_{u+5}\right) z_{u+5}^2$$

$$\text{(A19)}\quad = \tfrac{1}{2}\alpha_u\left(\omega z_u^2 - 2 z_u z_{u+5} + \omega^{-1} z_{u+5}^2\right) + z_u^2\left(\tfrac{1}{2}\alpha_{u-5} - \tfrac{1}{2}\alpha_u\omega\right) + z_{u+5}^2\left(\alpha_u - \tfrac{1}{2}\alpha_{u+5} - \tfrac{1}{2}\alpha_u\omega^{-1}\right).$$

The last expression is obtained from the preceding by adding and subtracting a quantity involving ω and a quantity involving ω^{-1} where ω at this point is an unspecified positive number. The last expression (A19) is of course equal to

$$\text{(A20)}\quad \tfrac{1}{2}\alpha_u\left(\sqrt{\omega} z_u - \sqrt{\omega^{-1}}\, z_{u+5}\right)^2 + z_u^2\left(\tfrac{1}{2}\alpha_{u-5} - \tfrac{1}{2}\alpha_u\omega\right) + z_{u+5}^2\left(\alpha_u - \tfrac{1}{2}\alpha_u\omega^{-1} - \tfrac{1}{2}\alpha_{u+5}\right).$$

The question now is whether or not it is possible to assign a value to ω such that the coefficients of z_u^2 and z_{u+5}^2 in (A20) are both positive. For the coefficient of z_u^2 to be positive it is necessary that $\alpha_{u-5} > \alpha_u\omega$, or

$$\text{(A21)}\quad \omega < \frac{\alpha_{u-5}}{\alpha_u}$$

For the coefficient of z_{u+5}^2 to be positive it is necessary that

$$\left(1 - \tfrac{1}{2}\omega^{-1}\right)\alpha_u > \tfrac{1}{2}\alpha_{u+5},$$

or

$$\text{(A22)}\quad \omega > \frac{\alpha_u}{2\alpha_u - \alpha_{u+5}}.$$

That both of the inequalities (A21) and (A22) are satisfied means that

$$\text{(A23)}\quad \frac{\alpha_u}{2\alpha_u - \alpha_{u+5}} < \omega < \frac{\alpha_{u-5}}{\alpha_u}.$$

In other words a value of ω can be found for which the coefficients of z_u^2 and z_{u+5}^2 in (A20) are both positive, provided

$$\text{(A24)}\quad \frac{\alpha_u}{2\alpha_u - \alpha_{u+5}} < \frac{\alpha_{u-5}}{\alpha_u}.$$

Inequality (A24) is equivalent to

$$\text{(A25)}\quad \frac{\alpha_u}{\alpha_{u-5}} + \frac{\alpha_{u+5}}{\alpha_u} < 2,$$

or

$$\text{(A26)}\quad \frac{{}_5m_{u-5}^{pop}}{{}_5m_u^{pop}} + \frac{{}_5m_u^{pop}}{{}_5m_{u+5}^{pop}} < 2.$$

Thus a second result is the following:

Theorem A2: The system of linear equations defined by equations (11), (12), and (13) has one and only one solution for the unknowns $\ell_{10}, \ell_{15}, \ldots, \ell_{105}$ if $\frac{1}{2}\,{}_5m_5^{pop} < {}_5m_{10}^{pop}$ and ${}_5m_x^{pop} < {}_5m_{x+5}^{pop}$, $x = 10, 15, \ldots, 90$ with the exception of one age u, $15 \leqq u \leqq 85$, for which ${}_5m_u^{pop} > {}_5m_{u+5}^{pop}$ provided

$$\frac{{}_5m_{u-5}^{pop}}{{}_5m_u^{pop}} + \frac{{}_5m_u^{pop}}{{}_5m_{u+5}^{pop}} < 2.$$

Thus, a ratio of ${}_5m_x^{pop}/{}_5m_{x+5}^{pop}$ greater than unity can be tolerated provided the corresponding ratio at the preceding age is sufficiently less than unity so that their sum is less than 2. In a very similar manner one can obtain a corresponding result in terms of a ratio ${}_5m_x^{pop}/{}_5m_{x+5}^{pop}$ and the following ratio. That result may be stated as follows:

Theorem A3: The system of linear equations defined by equations (11), (12), and (13) has one and only one solution for the unknowns $\ell_{10}, \ell_{15}, \ldots, \ell_{105}$ if $\frac{1}{2}\,{}_5m_5^{pop} < {}_5m_{10}^{pop}$ and ${}_5m_x^{pop} < {}_5m_{x+5}^{pop}$, $x=10,15,\ldots,90$ with the exception of one age $u, 10 \leqq u \leqq 80$, for which

$$ {}_5m_u^{pop} > {}_5m_{u+5}^{pop} $$

provided

$$ \frac{{}_5m_u^{pop}}{{}_5m_{u+5}^{pop}} + \frac{{}_5m_{u+5}^{pop}}{{}_5m_{u+10}^{pop}} < 2. $$

As an example the 1959-61 U.S. life table for white males is a case which is not covered by Theorem A1 but is covered by either Theorem A2 or Theorem A3. The following values are from that table.

x	${}_5m_x^{pop}$	${}_5m_x^{pop}/{}_5m_{x+5}^{pop}$
15	0.0012371	0.7367
20	0.0016793	1.1282
25	0.0014885	0.8616
30	0.0017275	

In this case the sums of both pairs of successive ratios are less than 2 (although one of them is 1.9898). It may well be that for some other table only one or the other of the sums is less than 2, in which case the uniqueness of the solution to the system of equations is established by appeal to only one of Theorems A2 and A3.

It should be noted that the conditions of these three theorems are by no means exhaustive of those under which a unique solution to the system of equations exists. For example ${}_5m_x^{pop} > {}_5m_{x+5}^{pop}$ could be permitted more than once in a single table provided that the occurrences were not too close together. Hence, if a table is encountered which satisfies the conditions of neither of Theorems A2 and A3, it by no means follows that a unique solution to the system of equations does not exist. A case in which there is not a unique solution would be rare indeed.

CONVERGENCE OF THE ITERATIVE PROCESS

This section is concerned with finding conditions under which the iterative process for determining life-table values defined by equations (16) and (17) converges.

The problem of convergence may be attacked in a number of ways. Letting a superscript r denote as before the life-table values obtained at the rth iteration, one may express equation (16) in the form

(B1) $$ {}_5f_x^r = \frac{5}{{}_5q_x^r} - \frac{1}{{}_5m_x^{pop}}, $$

and equation (17) (for $r>1$) in the form

(B2) $$ {}_5f_x^r = \frac{5}{{}_5q_x^{r-1}} - \frac{1}{{}_5m_x^{r-1}} $$

By subtracting equation (B2) from (B1), one obtains

(B3) $$ \frac{5}{{}_5q_x^r} - \frac{5}{{}_5q_x^{r-1}} = \frac{1}{{}_5m_x^{pop}} - \frac{1}{{}_5m_x^{r-1}} $$

Using equation (B1) and (B1) with r replaced by $r-1$ one also has that

(B4) $$ \frac{5}{{}_5q_x^r} - \frac{5}{{}_5q_x^{r-1}} = {}_5f_x^r - {}_5f_x^{r-1} . $$

From the latter two equations it is seen that convergence of the process in the sense that values of ${}_5q_x^r$ tend toward equality with increasing r is equivalent to the condition that values of ${}_5f_x^r$ similarly tend toward equality, and both of these are in turn equivalent to the condition that values of ${}_5m_x^r$ approach ${}_5m_x^{pop}$. In what follows convergence will be examined from the point of view of values of ${}_5m_x^r$ approaching ${}_5m_x^{pop}$ as the number of iterations increases. First it will be shown that the iterative process does converge provided certain relationships hold throughout the process, and then there will be determined conditions with respect to the values of ${}_5m_x^{pop}$ which guarantee that those relationships do in fact hold.

With $\alpha_x = \dfrac{1}{{}_5m_x^{pop}}$ and $\alpha_x^r = \dfrac{1}{{}_5m_x^r}$, let ϵ_x^r be defined as

(B5) $$\epsilon_x^r = {}_5L_x^r - \alpha_x({}_5d_x^r) = {}_5d_x^r(\alpha_x^r - \alpha_x).$$

It may be noted that if ϵ_x^r approaches zero as r increases, then α_x^r approaches α_x and ${}_5m_x^r$ approaches ${}_5m_x^{pop}$, provided ${}_5d_x^r$ does not approach zero.

Since equation (17) is equivalent to

(B6) $$(\alpha_x^{r-1} + {}_5f_x^r)\,{}_5d_x^{r-1} = 5\ell_x^{r-1},$$

or

(B7) $${}_5L_x^{r-1} = 5\ell_x^{r-1} - {}_5f_x^r({}_5d_x^{r-1}),$$

it follows that

(B8) $$\epsilon_x^{r-1} = 5\ell_x^{r-1} - (\alpha_x + {}_5f_x^r)\,{}_5d_x^{r-1}.$$

Also equation (B1) may be written as

(B9) $$(\alpha_x + {}_5f_x^r)\,{}_5d_x^r - 5\ell_x^r = 0,$$

so that

(B10) $$\begin{aligned}\epsilon_x^{r-1} &= 5\ell_x^{r-1} - (\alpha_x + {}_5f_x^r)\,{}_5d_x^{r-1} \\ &\quad - 5\ell_x^r + (\alpha_x + {}_5f_x^r)\,{}_5d_x^r \\ &= 5\ell_x^{r-1} - (\alpha_x + {}_5f_x^r)(\ell_x^{r-1} - \ell_{x+5}^{r-1}) \\ &\quad - 5\ell_x^r + (\alpha_x + {}_5f_x^r)(\ell_x^r - \ell_{x+5}^r).\end{aligned}$$

The last equation may be rearranged as

(B11) $$\begin{aligned}\epsilon_x^{r-1} &= (\alpha_x + {}_5f_x^r - 5)(\ell_x^r - \ell_x^{r-1}) \\ &\quad - (\alpha_x + {}_5f_x^r)(\ell_{x+5}^r - \ell_{x+5}^{r-1}),\end{aligned}$$

or

(B12) $$\begin{aligned}(\alpha_x + {}_5f_x^r)(\ell_{x+5}^r - \ell_{x+5}^{r-1}) &= -\epsilon_x^{r-1} \\ &\quad + (\alpha_x + {}_5f_x^r - 5)(\ell_x^r - \ell_x^{r-1}).\end{aligned}$$

Assuming that $\alpha_x + {}_5f_x^r \geq 5$ and employing a well known inequality for real numbers, one obtains that

(B13) $$\begin{aligned}(\alpha_x + {}_5f_x^r)|\ell_{x+5}^r - \ell_{x+5}^{r-1}| &\leq |\epsilon_x^{r-1}| \\ &\quad + (\alpha_x + {}_5f_x^r - 5)|\ell_x^r - \ell_x^{r-1}|.\end{aligned}$$

Next, let η_r denote the maximum value of $|\ell_x^r - \ell_x^{r-1}|$ for $x = 10, 15, \ldots, 95$, and assume that the maximum occurs at age $y + 5$. Then

$$\begin{aligned}(\alpha_y + {}_5f_y^r)\eta_r &\leq |\epsilon_y^{r-1}| + (\alpha_y + {}_5f_y^r - 5)|\ell_y^r - \ell_y^{r-1}| \\ &\leq |\epsilon_y^{r-1}| + (\alpha_y + {}_5f_y^r - 5)\eta_r .\end{aligned}$$

Hence, if we assume that $\alpha_y + {}_5f_y^r > 5$,

(B14) $$\eta_r \leq \tfrac{1}{5}|\epsilon_y^{r-1}| \leq \frac{1}{5} \max_{x=5,10,\ldots,90} |\epsilon_x^{r-1}|.$$

Inequality (B14) is the first of two inequalities which together will show the convergence of the process. To obtain the second, one uses equations (B7) and (B9) to produce

(B15) $$\begin{aligned}\epsilon_x^r &= {}_5L_x^r - {}_5L_x^{r-1} + {}_5f_x^r({}_5d_x^r - {}_5d_x^{r-1}) \\ &\quad - 5(\ell_x^r - \ell_x^{r-1}).\end{aligned}$$

Then by way of equations (8), (9), and (10) it follows that

(B16) $$\begin{aligned}\epsilon_5^r &= (2.623337 - {}_5f_5^r)(\ell_{10}^r - \ell_{10}^{r-1}) \\ &\quad - .300185(\ell_{15}^r - \ell_{15}^{r-1}) \\ &\quad + .037251(\ell_{20}^r - \ell_{20}^{r-1}) \\ &\quad + .000905(\ell_{25}^r - \ell_{25}^{r-1}),\end{aligned}$$

(B17) $$\begin{aligned}\epsilon_{10}^r &= ({}_5f_{10}^r - 2.151542)(\ell_{10}^r - \ell_{10}^{r-1}) \\ &\quad + (2.779262 - {}_5f_{10}^r)(\ell_{15}^r - \ell_{15}^{r-1}) \\ &\quad - .328328(\ell_{20}^r - \ell_{20}^{r-1}) \\ &\quad + .041967(\ell_{25}^r - \ell_{25}^{r-1}),\end{aligned}$$

and

$$\text{(B18)}\quad \epsilon^r_x = .0368\,(\ell^r_{x-10} - \ell^{r-1}_{x-10}) - .3104\,(\ell^r_{x-5} - \ell^{r-1}_{x-5})$$
$$+ ({}_5f^r_x - 2.2264)\,(\ell^r_x - \ell^{r-1}_x)$$
$$+ (2.7736 - {}_5f^r_x)\,(\ell^r_{x+5} - \ell^{r-1}_{x+5})$$
$$- .3104\,(\ell^r_{x+10} - \ell^{r-1}_{x+10})$$
$$+ .0368\,(\ell^r_{x+15} - \ell^{r-1}_{x+15}),$$

for $x = 15, 20, \ldots, 90$. It should be noted that ℓ_4 and ℓ_5 are assumed to be constant for all iterations, and that $\ell^r_4 - \ell^{r-1}_4$ and $\ell^r_5 - \ell^{r-1}_5$ are therefore equal to zero. If $1 < {}_5f^r_x < 4$, then $|{}_5f^r_{10} - 2.151542| + |2.779262 - {}_5f^r_{10}| < 3.07$ and $|{}_5f^r_x - 2.2264| + |2.7736 - {}_5f^r_x| < 3$, with the following three inequalities then resulting:

$$\text{(B19)}\quad |\epsilon^r_5| < 1.97\eta_r,\ |\epsilon^r_{10}| < 3.44\eta_r, \text{ and } |\epsilon^r_x| < 3.70\eta_r,$$

and the latter holding for $x = 15, 20, \ldots, 90$. Hence one may conclude that

$$\text{(B20)}\quad \max_{x=5,10,\ldots,90} |\epsilon^r_x| < 3.70\eta_r.$$

Combining (B14) and (B20) produces the inequality

$$\text{(B21)}\quad \max_{x=5,10,\ldots,90} |\epsilon^r_x| < .74 \max_{x=5,10,\ldots,90} |\epsilon^{r-1}_x|,$$

which shows that ϵ^r_x approaches zero as r increases and that the process converges, provided the three assumptions which have been made, (a) ${}_5d^r_x$ does not approach zero, (b) $\alpha_x + {}_5f^r_x \geqq 5$, and (c) $1 < {}_5f^r_x < 4$, do in fact hold for any life table under consideration.

To determine conditions such that the three assumptions hold, let r be replaced by $r+1$ in equation (B7). Solving for ${}_5f^{r+1}_x$ then produces

$$\text{(B22)}\quad {}_5f^{r+1}_x = \frac{5\ell^r_x - {}_5L^r_x}{{}_5d^r_x},$$

from which one may obtain, using equation (10), that

$$\text{(B23)}\quad {}_5f^{r+1}_x = 2.5 + \frac{-.0368\,{}_5d^r_{x-10} + .2736\,{}_5d^r_{x-5} - .2736\,{}_5d^r_{x+5} + .0368\,{}_5d^r_{x+10}}{{}_5d^r_x}$$

Dividing numerator and denominator of the second term on the right by ℓ^r_x produces

$$\text{(B24)}\quad {}_5f^{r+1}_x = 2.5 + \frac{-.0368\,{}_5q^r_{x-10}\dfrac{1}{1-{}_5q^r_{x-10}}\cdot\dfrac{1}{1-{}_5q^r_{x-5}} + .2736\,{}_5q^r_{x-5}\dfrac{1}{1-{}_5q^r_{x-5}}}{{}_5q^r_x}$$
$$+ \frac{-.2736\,(1-{}_5q^r_x)\,{}_5q^r_{x+5} + .0368\,(1-{}_5q^r_x)(1-{}_5q^r_{x+5})\,{}_5q^r_{x+10}}{{}_5q^r_x}.$$

The latter two equations are valid for $x = 15, 20, \ldots, 90$, with the understanding that ${}_5q^r_{95}$ and ${}_5q^r_{100}$ have values constant for all r based on Union Civil War veterans' mortality experience. For $x = 5$ and $x = 10$ the corresponding results using equations (8) and (9), respectively, are

$$\text{(B25)}\quad {}_5f^{r+1}_5 = 2.5 + \frac{\begin{aligned}\Big[&1.808303\,q_4\frac{1}{1-q_4} - .138692\,{}_5q^r_5 \\ &- .262029\,(1-{}_5q^r_5)\,{}_5q^r_{10} \\ &+ .038156\,(1-{}_5q^r_5)(1-{}_5q^r_{10})\,{}_5q^r_{15} \\ &+ .000905\,(1-{}_5q^r_5) \\ &(1-{}_5q^r_{10})(1-{}_5q^r_{15})\,{}_5q^r_{20}\Big]\end{aligned}}{{}_5q^r_5}$$

and

(B26)
$$
\Big[-.449328\, q_4 \cdot \frac{1}{1-q_4}\, \frac{1}{1-{}_5q_5^r} + .341359\, {}_5q_5^r\, \frac{1}{1-{}_5q_5^r} - .007099\, {}_5q_{10}^r - .286361(1-{}_5q_{10}^r)\, {}_5q_{15}^r + .041967\,(1-{}_5q_{10}^r)\, (1-{}_5q_{15}^r)\, {}_5q_{20}^r \Big]
$$
$$
{}_5f_{10}^{r+1} = 2.5 + \frac{\Big[\cdots\Big]}{{}_5q_{10}^r}
$$

What one would like to conclude is that if at the rth iteration $1 < {}_5f_x^r < 4$ and $\alpha_x + {}_5f_x^r \geqq 5$, then the same is true at the $(r+1)$st iteration, provided certain conditions hold among the values of α_x. Then, if the two conditions hold at the first iteration, it would follow by mathematical induction that they hold at all iterations, the proof of convergence thereby being established provided the proper conditions among the α_x hold. A similar but somewhat more complicated result will now be obtained.

Suppose there are numbers ${}_5f_x^L$ and ${}_5f_x^U$ such that $1 \leq {}_5f_x^L < {}_5f_x^U \leq 4$ and $\alpha_x + {}_5f_x^L > 5$. Suppose also that at the r^{th} iteration ${}_5f_x^L < {}_5f_x^r < {}_5f_x^U$ for $x = 5, 10, \ldots, 90$. Then, from equation (B1) it follows that

(B27) $$\frac{5}{\alpha_x + {}_5f_x^U} < {}_5q_x^r < \frac{5}{\alpha_x + {}_5f_x^L},$$

(B28) $$\frac{\alpha_x + {}_5f_x^L - 5}{\alpha_x + {}_5f_x^L} < 1 - {}_5q_x^r < \frac{\alpha_x + {}_5f_x^U - 5}{\alpha_x + {}_5f_x^U},$$

and

(B29) $$\frac{\alpha_x + {}_5f_x^U}{\alpha_x + {}_5f_x^U - 5} < \frac{1}{1 - {}_5q_x^r} < \frac{\alpha_x + {}_5f_x^L}{\alpha_x + {}_5f_x^L - 5}.$$

Employing these three inequalities in connection with equations (B24), (B25), and (B26), one may obtain inequalities of the form

(B30) $${}_5f_x^{r+1} > 2.5 - \phi\ (x),$$

and

(B31) $${}_5f_x^{r+1} < 2.5 + \psi\ (x),$$

for $x = 5, 10, \ldots, 90$, where $\phi(x)$ and $\psi(x)$ are given by

$$\phi\ (5) = .262029\ \frac{\alpha_5 + {}_5f_5^U - 5}{\alpha_{10} + {}_5f_{10}^L} + .138692,$$

$$\phi\ (10) = .0898656\ \frac{q_4}{p_4}\ \frac{\alpha_5 + {}_5f_5^L}{\alpha_5 + {}_5f_5^L - 5}\ (\alpha_{10} + {}_5f_{10}^U) + .286361\ \frac{\alpha_{10} + {}_5f_{10}^U - 5}{\alpha_{15} + {}_5f_{15}^L} + .007099,$$

$$\phi\ (x) = .0368\ \frac{(\alpha_{x-5} + {}_5f_{x-5}^L)(\alpha_x + {}_5f_x^U)}{(\alpha_{x-10} + {}_5f_{x-10}^L - 5)(\alpha_{x-5} + {}_5f_{x-5}^L - 5)} + .2736\ \frac{\alpha_x + {}_5f_x^U - 5}{\alpha_{x+5} + {}_5f_{x+5}^L},$$

for $x = 15, 20, \ldots, 85$,

$$\phi\ (90) = .0368\ \frac{(\alpha_{85} + {}_5f_{85}^L)(\alpha_{90} + {}_5f_{90}^U)}{(\alpha_{80} + {}_5f_{80}^L - 5)(\alpha_{85} + {}_5f_{85}^L - 5)} + .05472\ {}_5q_{95}\ (\alpha_{90} + {}_5f_{90}^U - 5),$$

$$\psi\ (5) = .3616606\ \frac{q_4}{p_4}\ (\alpha_5 + {}_5f_5^U) + .038156\ \frac{\alpha_5 + {}_5f_5^U - 5}{\alpha_{15} + {}_5f_{15}^L} + .000905\ \frac{\alpha_5 + {}_5f_5^U - 5}{\alpha_{20} + {}_5f_{20}^L} - .138692,$$

$$\psi\ (10) = .341359\ \frac{\alpha_{10} + {}_5f_{10}^U}{\alpha_5 + {}_5f_5^L - 5} + .041967\ \frac{\alpha_{10} + {}_5f_{10}^U - 5}{\alpha_{20} + {}_5f_{20}^L} - .007099,$$

$$\psi(x) = .2736\ \frac{\alpha_x + {}_5f_x^U}{\alpha_{x-5} + {}_5f_{x-5}^L - 5} + .0368\ \frac{\alpha_x + {}_5f_x^U - 5}{\alpha_{x+10} + {}_5f_{x+10}^L},$$

for $x = 15, 20, \ldots, 80,$

$$\psi(85) = .2736\ \frac{\alpha_{85} + {}_5f_{85}^U}{\alpha_{80} + {}_5f_{80}^L - 5} + .00736\ {}_5q_{95}(\alpha_{85} + {}_5f_{85}^U - 5),$$

$$\psi(90) = .2736\ \frac{\alpha_{90} + {}_5f_{90}^U}{\alpha_{85} + {}_5f_{85}^L - 5}$$

$$+ .00736\ (1 - {}_5q_{95})\ {}_5q_{100}\ (\alpha_{90} + {}_5f_{90}^U - 5).$$

The inductive step will be complete if the values of $\phi(x)$ and $\psi(x)$ are such that ${}_5f_x^L < {}_5f_x^{r+1} < {}_5f_x^U$. This will certainly be the case if ${}_5f_x^L \leq 2.5 - \phi(x)$ and $2.5 + \psi(x) \leq {}_5f_x^U$, or equivalently

(B32) $\quad \phi(x) \leq 2.5 - {}_5f_x^L,$

and

(B33) $\quad \psi(x) \leq {}_5f_x^U - 2.5.$

The foregoing observations lead to a theorem which may be stated as follows:

Theorem: If there exist numbers ${}_5f_x^L$ and ${}_5f_x^U$, $x = 5, 10, \ldots, 90$, such that(i) $1 \leq {}_5f_x^L < {}_5f_x^U \leq 4$, (ii) $\alpha_x + {}_5f_x^L > 5$, (iii) $\phi(x) \leq 2.5 - {}_5f_x^L$, and (iv) $\psi(x) \leq {}_5f_x^U - 2.5$, then the iterative process converges.

Proof: If values of ${}_5f_x^L$ and ${}_5f_x^U$ exist which satisfy the four inequalities, then $\phi(x) > 0$ and $\psi(x) > 0$ and ${}_5f_x^L < 2.5 < {}_5f_x^U$. Hence ${}_5f_x^L < {}_5f_x^1 < {}_5f_x^U$ because ${}_5f_x^1 = 2.5$. Also if ${}_5f_x^L < {}_5f_x^r < {}_5f_x^U$, then from inequalities (B30) and (B31) and conditions (iii) and (iv) it follows that ${}_5f_x^L < {}_5f_x^{r+1} < {}_5f_x^U$. Hence by mathematical induction one has that ${}_5f_x^L < {}_5f_x^r < {}_5f_x^U$ for all positive integers r, and consequently $1 < {}_5f_x^r < 4$ and $\alpha_x + {}_5f_x^r > 5$ for all r.

All that remains to be shown is that ${}_5d_x^r$ does not approach zero as r increases. However

$${}_5d_x^r = \ell_5\,(1 - {}_5q_5^r)(1 - {}_5q_{10}^r) \ldots (1 - {}_5q_{x-5}^r)\ {}_5q_x^r.$$

By virtue of inequalities (B27) and (B28) one may conclude, therefore, that

$$\text{(B34)}\quad {}_5d_x^r > \ell_5 \left(\frac{\alpha_5 + {}_5f_5^L - 5}{\alpha_5 + {}_5f_5^L}\right)\left(\frac{\alpha_{10} + {}_5f_{10}^L - 5}{\alpha_{10} + {}_5f_{10}^L}\right) \ldots \left(\frac{\alpha_{x-5} + {}_5f_{x-5}^L - 5}{\alpha_{x-5} + {}_5f_{x-5}^L}\right) \frac{5}{\alpha_x + {}_5f_x^U},$$

which shows that the limiting value of ${}_5d_x^r$ as r increases is at least as large as the expression on the right side of inequality (B33) and is therefore not zero under the hypotheses of this theorem.

A way of using the theorem to show that the iterative process does converge for a particular table is to proceed as follows: First, set ${}_5f_x^U = 4$ for all x. Second, set ${}_5f_x^L = 1$ for those ages x for which $\alpha_x > 4$. If $2.5 < \alpha_x \leq 4$, set ${}_5f_x^L$ such that $\alpha_x + {}_5f_x^L > 5$. For example one might try initially ${}_5f_x^L = 5.1 - \alpha_x$. (If $\alpha_x \leq 2.5$, i.e., ${}_5m_x^{pop} \geq .4$, then the theorem is of no value in showing convergence of the process, even though the process may actually converge.) Third, compute $\phi(x)$ and $\psi(x)$ for all ages x. If the values of $\phi(x)$ and $\psi(x)$ obtained satisfy the inequalities $\phi(x) \leq 2.5 - {}_5f_x^L$ and $\psi(x) \leq {}_5f_x^U - 2.5$, then convergence is proved. If the inequalities are not satisfied, the values chosen for ${}_5f_x^L$ and ${}_5f_x^U$ may be changed and revised values of $\phi(x)$ and $\psi(x)$ obtained. In particular the values of ${}_5f_x^L$ at ages 85 and 90 very likely need to be increased. For example for the life table for United States white males with ${}_5f_x^U = 4$ for all ages x and ${}_5f_x^L = 1$ for $x = 5, 10, \ldots, 85$ and ${}_5f_{90}^L = 1.7$ $(\alpha_{90} = 3.41)$, one finds that the values of $\phi(x)$ and $\psi(x)$ which are obtained satisfy the inequalities $\phi(x) \leq 2.5 - {}_5f_x^L$ and $\psi(x) \leq {}_5f_x^U - 2.5$ except at age 90 where $\psi(90) = 1.71$.

However after changing ${}_5f_{85}^L$ to 1.8, one obtains a revised value for $\psi(90)$ of 1.02. The values of $\phi(80)$, $\phi(90)$, and $\psi(75)$ also change, but only slightly. At this stage both of the inequalities are satisfied at all ages and convergence has been proved.

Thus, a way of establishing that the iterative process does converge is provided by the theorem. However, as stated previously, inability to find values of ${}_5f_x^L$ and ${}_5f_x^U$ which satisfy the hypotheses of the theorem by no means implies that the process does not converge. It appears unlikely that a table can be found for which the process does not converge.

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