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CHARACTERIZATION OF THE STRUCTURAL BEHAVIOR OF ROCK MASSES

VOLUME 1

by

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and
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College of Engineering
Department of Civil Engineering

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This report was prepared by the University of California--Davis, College of Engineering, Department of Civil Engineering at Davis, California under USBM Grant Number G0133122. The contract was initiated under the Advancing Mining Technology/Metal and Nonmetal Mining Technology Program. It was administered under the technical direction of the Denver Mining Research Center with Mr. R. D. Caudle acting as Technical Project Officer. Mr. J. C. Edwards was the contract administrator for the Bureau of Mines.

This report is a summary of work recently completed as part of this contract during the period 06/26/73 to 08/26/74. This report was submitted by the authors on 12/30/74.

I ABSTRACT

The research described in this report is the first phase of a comprehensive program for the development of a characterization for the structural properties of rock masses. The mathematical expression of the characterization is especially tailored for use in conjunction with modern structural analysis procedures. The objective of the first phase of the project is to determine the feasibility of representing the structural properties of rock masses in terms of the analysis of an appropriate "representative volume"; the results of this investigation are described in detail in the body of the report.

The current state-of-the-art concerning the understanding and description of the fundamental mechanisms of rock behavior are determined from a survey of existing literature. The development of a representative volume that captures the most important of these mechanisms is described. In order not to unduly obscure the fundamental aspects of this development, the mathematical analysis of the representative volume is limited to plane strain conditions.

A subroutine for the numerical evaluation of the proposed model (i.e., the analysis of the representative volume) is presented. The steps necessary for the incorporation of the subroutine directly into existing two-dimensional finite element structural analysis programs are described.

Comparisons between the predictions of the proposed characterization and published results of simple laboratory tests

are presented and discussed. Finally recommendations are given concerning suggested improvements of the model and the extension of it to the general three-dimensional case.

II INTRODUCTION

A. MOTIVATION AND SCOPE OF STUDY:

The assessment of the degree of safety, the environmental impact, and the effectiveness of proposed and existing mining operations often require a quantitative understanding of the structural behavior of certain rock configurations. In such studies it is either the reliability over and against rock failure (safety and environmental impact studies) or the precipitation of rock failure (studies of the feasibility of proposed mining operations, e.g., caving) which is of interest. Hence, when the term "structural analysis" is used, the prediction of structural behavior out to and including failure, is meant.

The successful performance of a structural analysis requires:

(a) physical and mathematical descriptions of the geometry of the configuration and the environmental history to which it is subjected, e.g., loads, temperature, sequence of mining operations, etc.,

(b) mathematical models for the structural response and failure characteristics of the constituent materials (e.g., rocks, shoring, etc.), and

(c) an analysis procedure that is capable of predicting the behavior of the structure given the information from the above two items.

A consideration of the first item is beyond the scope of the present study. The development of the finite element structural analysis procedure has gone a long way towards providing a general analysis tool (satisfaction of the third item). Unfortunately, for many rock structures, the capabilities and potential of the finite element method can not be fully realized because of the deficiencies in the mathematical descriptions of the structural response and failure characteristics of the constituent materials; it is to this problem that this study is addressed.

There have, of course, been numerous experimental and theoretical studies conducted which relate to the problem of the characterization of the structural properties of rock (references to a number of these studies are included in other sections of this report). Unfortunately, these studies have not lead to the comprehensive description of the material properties of rock masses that is required for most structural analyses of mine related rock structures. The inadequacy of available characterizations is well stated in the following quotes from Howe-73* and Brace-64, i.e., "Yule marble....does not conform to any published constitutive equation cited in the literature to date" and "After trying all possible failure or yield conditions for metals, as well as failure conditions originally conceived for brittle materials, most investigators have concluded that no existing failure law holds for rocks in general, or even for a single rock under different conditions of loading." The seeming inadequacy of the experimental information, and the mathematical characterizations generated to date, are not due to the ineptness of the investigators but rather due to the exceptional complexity of the problem. Most

* The references referred to in the main body of the report are listed in Section VII. The references are listed by the first author's last name followed by a hyphen and the year of publication.

past studies were of necessity limited in scope to the consideration of only a few aspects of the overall problem, and in addition many of the studies were conducted before a full appreciation was developed for the extent and completeness of the material characterization needed for modern structural analysis purposes.

The purpose of the present study is to initiate a program whose ultimate goal is the development of a characterization of the structural response and failure properties of rock masses that may be directly incorporated into available advanced finite element analysis procedures; the results to date of this research are described in the main body of this report.

A secondary and independent phase of this project is a study of the fundamental mechanisms of rock behavior in terms of the response of a particular mathematical model. This research represents a portion of the doctoral research of one of the student Research Assistants employed on the project. A summary of this phase of the research, given in Appendix C, was written independently of the main body of the report. Thus there is no cross referencing and there may be some duplication. When the thesis is completed copies will be sent to the Bureau of Mines.

B. FORM AND SCOPE OF DESIRED CHARACTERIZATION:

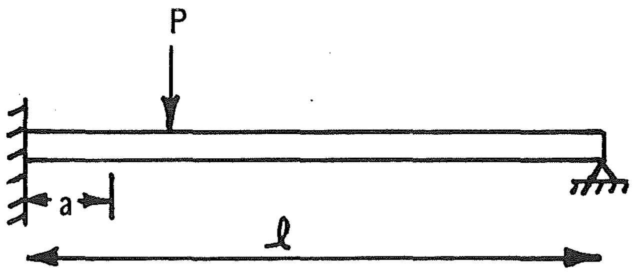
The required form and scope of a characterization of the structural properties of rock is determined by the physical situations to be analyzed and the analysis procedure to be used. The characterization must accurately predict the structural behavior of the rock for all stress and strain histories that will be experienced by all parts of the several rock structures to be analyzed. Such a precise definition of the limits of material behavior for which the characterization needs to be valid are impossible to determine (if prior to performance of an analysis the stress and strain histories were known there would, of course, be no need for the analysis). Thus practically what must be done is to:

(a) provide a characterization for all anticipated stress and strain histories, and

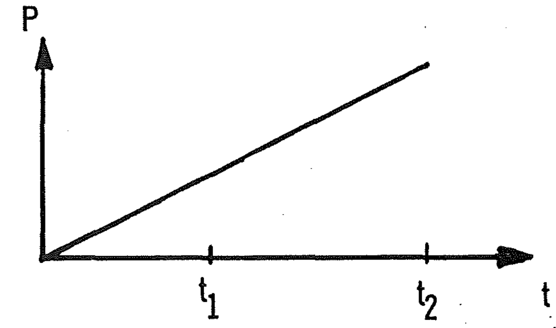
(b) be prepared to extend the characterization and repeat the analysis if certain stress and strain histories are predicted that were not accounted for by the original characterization.

When possible, in order to avoid the repetitive effort suggested by "b", a more general characterization than suggested by "a" should be initially provided. Because of the near impossibility of trying to anticipate all possible stress and strain histories which might arise in the analysis of a given complicated configuration, and because of the equally difficult task of trying to anticipate all possible future applications of the characterization, it is desirable, if possible, to have the characterization yield reasonable and consistent results for all stress and strain histories. Thus in the development initiated in this study, it is required of the characterization that it should yield reasonable predictions for all stress and strain histories. Due to the absence of experimental evidence it is, however, recognized that for certain states the characterization will at best represent extrapolations based upon an intuitive appreciation and understanding of the behavior of rock masses, and thus for these situations only qualitatively correct results can be expected. Fortunately the preponderance of experimental evidence has been gathered for those states which most frequently occur in real situations, and thus the characterization should give good quantitative agreement for the dominant stress and strain states.

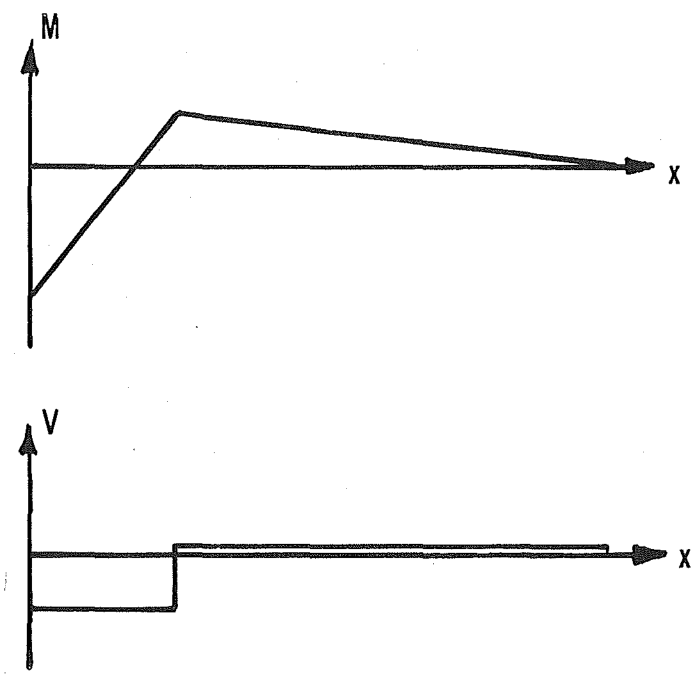
The importance of accounting for multi-axial stress and strain states, and for loading, unloading, reloading and non-proportionate states needs to be emphasized. The multi-axial nature (in contrast to the simple states often used in laboratory tests) of the stress and strain states that occur in complicated structural configurations is obvious. Less obvious, however, is the fact that the internal stresses may experience unloading, reloading and non-proportionate loading histories even though such is not the case for the external loads. This fact can be easily illustrated by a simple example. Consider the "elastic-perfectly plastic" beam (rectangular cross-section) shown in Figure 1a; it is to be noted that at time t_1 yielding begins at the left-hand support, and at time t_2 two plastic hinges have developed. Comparing Figures 1b and 1e, it is seen that although the external load is monotonically increasing some internal material elements successively experience loading, unloading and load reversal. In addition, comparing Figures 1e and 1f it is seen that, although the external loading is proportionate for some points within the body, the ratio of the internal stress components change during the loading history and thus the principal stress directions change. It is thus apparent, that for all but the simplest structures, that if such phenomena as nonlinear properties, local failure (e.g., cracking),



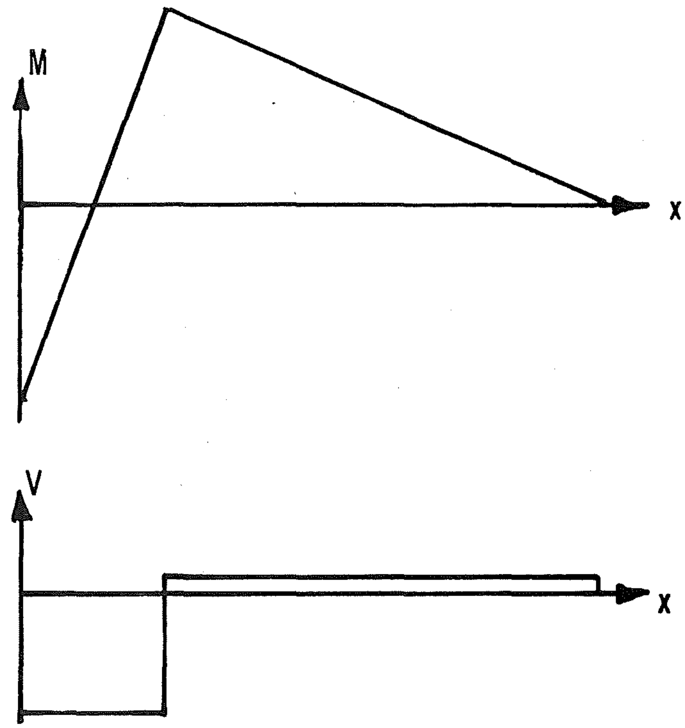
1_a Beam Configuration



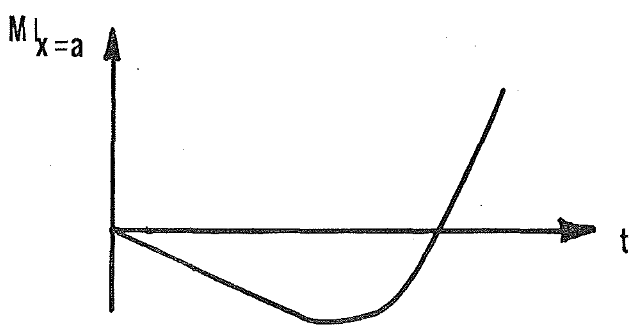
1_b History of Load



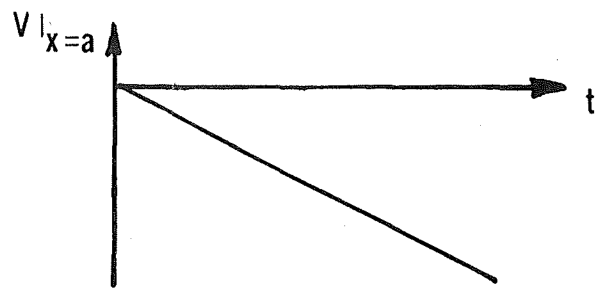
1_c Moment and Shear at t_1



1_d Moment and Shear at t_2



1_e History of Moment at x = a



1_f History of Shear at x = a

Figure 1 Simple Example To Illustrate Complex Stress History.

yielding, etc., occur, that even though the history of the applied loading may be simple, the history of the internal stresses and strains will usually be very complex.

The most convenient form in which to express a material characterization for use in conjunction with modern analysis procedures is now described. For a particular spatial location in the structure and a particular point in the time history (t_{N-1}), denote the stress and strain components as $[\sigma]_{N-1}$ and $[\varepsilon]_{N-1}$ (for three-dimensions $[\sigma]$ has the six components σ_x, σ_y, σ_z, τ_{xy}, τ_{xz}, τ_{yz})*. Denote the change in the strain during the next increment of time Δt_N (i.e., $t_N = t_{N-1} + \Delta t_N$) as $[\Delta\varepsilon]_N$; the accompanying change in the stress state $[\Delta\sigma]_N$ is expressed in the form:

$$[\Delta\sigma]_N = [C]_N \, [\Delta\varepsilon]_N + [L]_N \qquad (1)$$

In general, the matrices $[C]_N$ and $[L]_N$ are not only functions of $[\sigma]$ and $[\varepsilon]$ for $t = 0 \rightarrow t_{N-1}$ but also of $[\Delta\varepsilon]_N$ and $[\Delta\sigma]_N$ and thus will need to be established by iteration. It is to be noted that even though the material may be initially isotropic (and thus several of the components in $[C]_1$ are zero, etc.) for subsequent increments the material, in general, exhibits anisotropic incremental properties due to damage induced anisotropy. The presence of non-zero terms for all the coefficients of the $[C]_N$ matrix, the presence of the $[L]_N$

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* Throughout this development the stress components are limited to the usual components, i.e., normal and shear components; that is multi-polar components (e.g., couple stresses) are ignored. It is, however, the authors' opinion that they may be significant for some composite materials; their inclusion in modern structural analysis procedures is, however, not sufficiently advanced to warrant their consideration at this time.

matrix, and the fact that these matrices will, in general, vary from point to point within the body (even though the body might have been initially homogeneous) offers no particular difficulties for general nonlinear finite element analysis procedures. One feature which can, however, lead to difficulties is that after the initial loading stages (i.e., once the material begins to experience damage) the $[C]_N$ matrix may not be symmetric. Because, in general, finite element procedures for structural problems have been based upon the assumed symmetry of $[C]_N$, and because the modifications necessary to permit $[C]_N$ to be non-symmetric requires some effort and leads to a very considerable increase in computational cost, special consideration is now given to this phenomenon.

The matrix $[C]_N$ of equ. (1) may be written in the form:

$$[C]_N = [C_s]_N + [C_u]_N$$

Where the two matrices on the right are defined as follows (the T denotes matrix transpose):

$$[C_s]_N = \frac{1}{2}([C]_N + [C]_N^T)$$

$$[C_u]_N = \frac{1}{2}([C]_N - [C]_N^T)$$

Now equ. (1) is written in the form*:

$$[\Delta\sigma]_N = [C_s]_N\,[\Delta\varepsilon]_N + [L_s]_N \qquad (2)$$

* If it is desired that the incremental properties be "positive definite" a similar modification may be performed to insure that such be the case. In other inelastic studies (e.g., see Hossain-74) it has been found that such a step may greatly improve the convergence characteristics.

Where

$$[L_s]_N = [L]_N + [C_u]_N [\Delta\varepsilon]_N$$

It is to be recalled that, in general, $[C]_N$ and $[L]_N$ must be established in an iterative fashion; if one is dealing with iteration "m" of this procedure then the value of $[\Delta\varepsilon]_N$ used in calculating $[L_s]_N$ is the strain increment estimated from the m-1 iteration. The use for analysis purposes of equ. (2) in place of equ. (1) means, of course, that one can continue to take advantage of symmetry in solving the set of simultaneous equations that result from the finite element method; the disadvantage is that the convergence of the iteration procedure is slowed; however the overall computational efficiency of the program should be improved (as compared to solving the non-symmetric equations directly).

C. OUTLINE OF CHARACTERIZATION PROCEDURE:

The development of a characterization of the structural properties of rock proceeds from the following assessment of the current state of the art:

(a) There exists a very large quantity of published experimental evidence concerning the structural behavior of rock and, based upon this experimental evidence, a number of theories have been advanced to explain the various fundamental mechanisms involved in rock behavior. While this evidence and the resulting theories are sometimes contradictory and do not cover many important stress and strain histories, they can be used to construct a fairly acceptable picture of the phenomenalogical aspects of rock behavior.

(b) Based upon the above mentioned evidence a few mathematical characterizations of rock behavior have been proposed. It is the authors' judgment, however, that none of the characterizations proposed to date are adequate. In the authors' opinion, this inadequacy is due to the fact that they were not constructed with the needs of modern finite element procedures in mind, and thus, have not yielded characterizations which are valid and reasonable for many stress and strain histories encountered in finite element analyses of complicated rock structures.

It is the authors' opinion that the development of a general characterization of rock behavior should proceed as follows:

(a) From a careful review of existing experimental evidence, isolate the dominant characteristics of rock behavior.

(b) For each of the characteristics identified in "a", select from the several proposed theoretical explanations the one which appears to be most substantiated by experimental evidence.

(c) Develop a mathematical model based upon the fundamental mechanisms described in the previous step, and which yields consistent and reasonable behavior for those states for which experimental evidence is lacking. Care must be taken that this mathematical description is consistent with the laws of mechanics.

(d) Express the characterization developed in the previous step in such a form that it may, with little difficulty, be directly incorporated into more advanced finite element structural analysis programs.

(e) Utilize the results of the previous step, in conjunction with an existing finite element program, for the analysis of rock structures for which experimental measurements are available. Such comparisons might lead to the recognition of shortcomings and inaccuracies in the characterization which would then be rectified.

Steps (a), (b) and (c) will most surely bring to light many areas for which additional experimental evidence is needed. In addition, steps (c) and (d) will point out theoretical and mathematical deficiencies which need to be remedied. The final step of this procedure would be to suggest experimental and theoretical programs to remove these deficiencies.

In the remainder of this report the progress made towards the satisfaction of the above outlined goal is described.

III CHARACTERIZATION OF THE STRUCTURAL PROPERTIES OF ROCK

When discussing the material properties of rock, the level of observation must be carefully specified, i.e., properties for several different levels of observation can be distinguished. The levels of interest for this study are:

(a) microscopic - the level of observation that considers individual grains, microscopic cracks, voids, etc.,

(b) macroscopic - the level of observation concerned with small laboratory samples, and

(c) structural - the level of observation concerned with the gross structural properties of relatively large rock masses.

Because at times, one is required to perform structural analyses of laboratory samples, a clear distinction can not always be made between the last two categories.

A. DOMINANT CHARACTERISTICS OF ROCK BEHAVIOR

1. Response and Failure Mechanisms*:

Initial state: In the analysis of a rock structure, interest is focused upon how the structural material (i.e., the rock as it exists at the beginning of the period covered by the analysis) responds to the environmental changes imposed upon the structure (e.g., tunneling of neighboring rock, etc.). Thus, a distinction must be made between

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* The papers referred to in this section are not intended to give a complete listing of the available literature on any one phenomenon but rather represent the references that the authors found to be particularly enlightening; additional references are to be found in Appendix E.

damage (e.g., cracks) induced in the rock prior to the period of time considered by the analysis, and the damage induced during the period accounted for by the analysis (this period is called the "service period"). In theory, it is possible to eliminate, to some degree, this rather artificial distinction by extending the analysis back to include the geological history of the rock, but in practice this is not feasible.

Thus, what is called the initial or virgin state of the rock is usually the end result of a long and complicated history of stress and strain; thus, the importance and necessity of being able to describe the initial state of the material (including the stress, strain and damage states) is apparent.

Porosity and discontinuities: There appears to be three distinct types of discontinuities associated with rock masses and/or samples, i.e.,

(a) a system of small, generally, spherical voids existing since the formation of the rock, hereinafter called pores,

(b) microscopic cracks, often penny-like, which open when a rock sample is quarried from the parent formation (due to the relieving of the naturally existing compressive stresses) and,

(c) macroscopic failure or weakness planes (these planes will be discussed in the following section).

For rock samples the second type of porosity appears to be, in general, larger than the first, it however disappears rapidly upon compressive loading. These penny-like cracks are the probable cause of the nonlinearities observed in the initial stages of loading of laboratory samples (Handin-63, Brace-64, Murrell-65, Walsh-65, Brady-70); their closing appears to be essentially an elastic phenomenon. Because opening of the microscopic cracks occurs during the removal of samples from the parent rock, the contribution of their closures to the behavior of intact rock is questionable (Walsh-65, Bieniawski-67_b).

The first two types of discontinuities are, in general, distributed and oriented in a rather homogeneous isotropic manner (Walsh-65).

Macroscopic and structural planes of weakness and fracture: Most in situ rock is permeated by systems of approximately parallel natural (i.e., pre-existing) planes of weakness or fracture. Such systems may be due to bedding, or stress induced fracture caused by cooling, faulting, folding, etc. For a given rock, the number of such systems appears to seldom exceed four (often there are three approximately mutually perpendicular sets) (Pomeroy-71). In addition, stresses induced during the "service period" of a rock structure (due to loads, etc.) may produce additional systems of fracture planes.

The natural fracture planes may initially be open, filled (with some foreign material, e.g., clay), or closed; their initial state will, of course, highly influence their subsequent behavior, see Goodman-72 for an excellent discussion of this topic. Open and filled joints contribute to the behavior of the rock in an anisotropic manner, i.e., their effects are highly dependent upon the direction of applied loads. When stress induced sliding takes place along a closed joint system dilatation also takes place, i.e., the joints tend to separate. This separation appears to be caused by the riding of local asperities over one another. Due to differences in initial smoothness, this dilatation appears to be more pronounced for tensile induced failure planes than shear induced fracture planes. The magnitude of the dilatation is also dependent upon the magnitude of the normal stress which acts across the sliding planes (a large normal stress tending to shear off the asperities and reduce the friction).

The post maximum strength region of rock behavior would appear to be a region of large scale sliding and/or separation occuring along well established fracture planes. Closed fracture planes continue to exhibit a stiffness which appears to be a result of sliding friction (Jaeger-60, Herget-70, Hobbs-70), and thus the rock exhibits a residual strength even though "local fracture" has taken place (Hobbs-70). Continued movement along fracture planes appears to have a modifying effect upon the values of the friction and cohesion parameters.

Planes of initial weakness require a certain induced stress state before they fracture, subsequently they can be classified as fracture planes.

Microscopic cracks: As was noted previously (see comment on porosity and discontinuities) most in situ rock is permeated by microscopic cracks (normally closed for in situ conditions). As additional stress (this stress may add to or subtract from the in situ stress) is imposed upon the rock, the behavior of these cracks, their growth, and the formation of additional cracks appear to be major determining factors in the phenomenological behavior of rock (Bieniawski-67, Brady-69, Brady-70).

It has been suggested that relative sliding of opposing faces of microscopic cracks accounts for the small hysteresis loops often observed in, what otherwise appears to be an essentially linear elastic response for rock (i.e., when the stresses are substantially less than the ultimate strength of the rock) (Walsh-65, Bieniawski-67_b). The formation of new cracks, the stable propagation of existing cracks (i.e., requiring energy input from surroundings) and the linking of cracks would appear to account for the highly nonlinear inelastic response in the region near the ultimate strength of the rock. Because crack propagation is a time related phenomenon, the behavior of the rock in this region is rate dependent, and thus the ultimate strength may be highly rate dependent (Bieniawski-67_c, Brady-69). Because the principal stress directions influence the orientations of new cracks developed during this phase of behavior, damage induced anisotropy is experienced. This phase of rock behavior marks a transition between microscopic cracks and a macroscopic system (i.e., a stress induced macroscopic system which is in addition to any pre-existing systems) (Handin-63, Brace-64, Bieniawski-67_b). As relative motions of the opposing crack faces take place (sliding or separation) the cracks dilate, and thus the material tends to dilate (positive increase in volume) rapidly even though the mean pressure (average of principal stresses) may be compressive (Handin-63, Brace-64, Goodman-72). The explanation for the sliding induced dilatation of closed cracks is given in the previous section.

Unstable (requires no energy input from its surrounding) propagation and linking up of cracks would appear to result in the catastrophically rapid loss in strength which is observed beyond the ultimate load carrying capacity region.

Finally there is evidence that if the mean pressure is high enough the rapid crack propagation process is arrested in favor of a ductile type of dislocation behavior (Handin-63, Murrell-65, Bieniawski-67).

Initial orthotropic properties: Due to the non-isotropic conditions that often prevail during their formation (i.e., deposition of sedimentary rocks, unequal stresses during the formation of metamorphic rocks, etc.), rocks may possess initially orthotropic microscopic material properties. This initial orthotropy is to be distinguished from that induced by subsequent damage or the large scale orthotropy resulting from macroscopic or structural planes of weakness or fracture (Jaeger-60).

Pore pressure: Water contained within the pores of rocks can have a significant effect upon structural response characteristics. When this water is under pressure it effectively reduces the magnitude of the normal stresses by the value of the fluid pressure (Handin-63, Murrell-65).

Evidence concerning whether or not the presence of water has an effect upon the value of friction for fracture and weakness planes appears to be somewhat contradictory (Handin-63, Goodman-70).

Material variability: (Evans-58, Brace-64, Brady-69, Yamaguchi-70, Kostak-71) Due to the nature of the formation process and their composite nature, rocks are highly variable (i.e., inhomogeneous) materials. Inhomogeneities which occur on a very large scale may be accounted for by treating the rock as inhomogeneous at the structural level; inhomogeneities which occur on a small scale must, however, be considered as part of the intrinsic material properties. As a result of microscopic and macroscopic variability, long before the final structural failure of rock, it is permeated with local failures (e.g., cracks).

In addition, as a consequence of material variability rocks exhibit "size effects" (Evans-58, Brady-70, Hoagland-73) which must be accounted for when attempting to correlate the results of laboratory and field tests (in laboratory studies the variability is often suppressed by carefully selecting samples from the most sound regions of rock). The importance of accounting for size effects when describing material properties to be used in finite element analyses with variable element sizes is not well understood and needs further study.

Fragmented phase: During the fracturing process a certain amount of pulverized material is formed (Handin-63, Brace-64) in the crack. Initially open cracks often contain foreign material (e.g., soil) which may be treated as an initial fragmented phase.

Temperature dependence: In general, temperature affects the structural properties of rock tending to enhance ductility and reduce peak strength (Murrell-65). For many rocks these effects are small and relatively unimportant for temperatures below 300°C.

2. Some Qualitative Observations of Structural Response Characteristics:

The structural behavior of a rock mass is governed by the interactive response of pre-existing planes of weakness and fracture, and the macroscopic properties of the rock itself.

A qualitative discussion of the behavior of pre-existing planes of weakness and fracture is given in the previous section.

The structural behavior of rock (excluding pre-existing planes of weakness and fracture) can be qualitatively described in terms of several zones of behavior (a similar description for concrete is given by Romstad-74). These zones are illustrated for a simple uniaxial test of marble given by Wawersik-70, see Figure 2; it must be remembered that uniaxial stress-strain behavior is merely one small part of the overall spectrum of behavior. Zone I is very nearly elastic and is nearly linear for uniaxial tension. It is nonlinear for small uniaxial compressive stresses followed by an essentially linear region (the nonlinearity, for small compressive stress states, is due to the closing of pre-existing microscopic cracks as described earlier). Although rock is nearly elastic in this zone, unloading does exhibit a small hysteresis loop which is attributable to a certain amount of frictional sliding between opposing faces of the pre-existing cracks. There appears to be little damage done by loading in this zone and thus repeated unloading and reloading exhibit little deviation from the original loading path.

All zones beyond the first are characterized by a tendency for

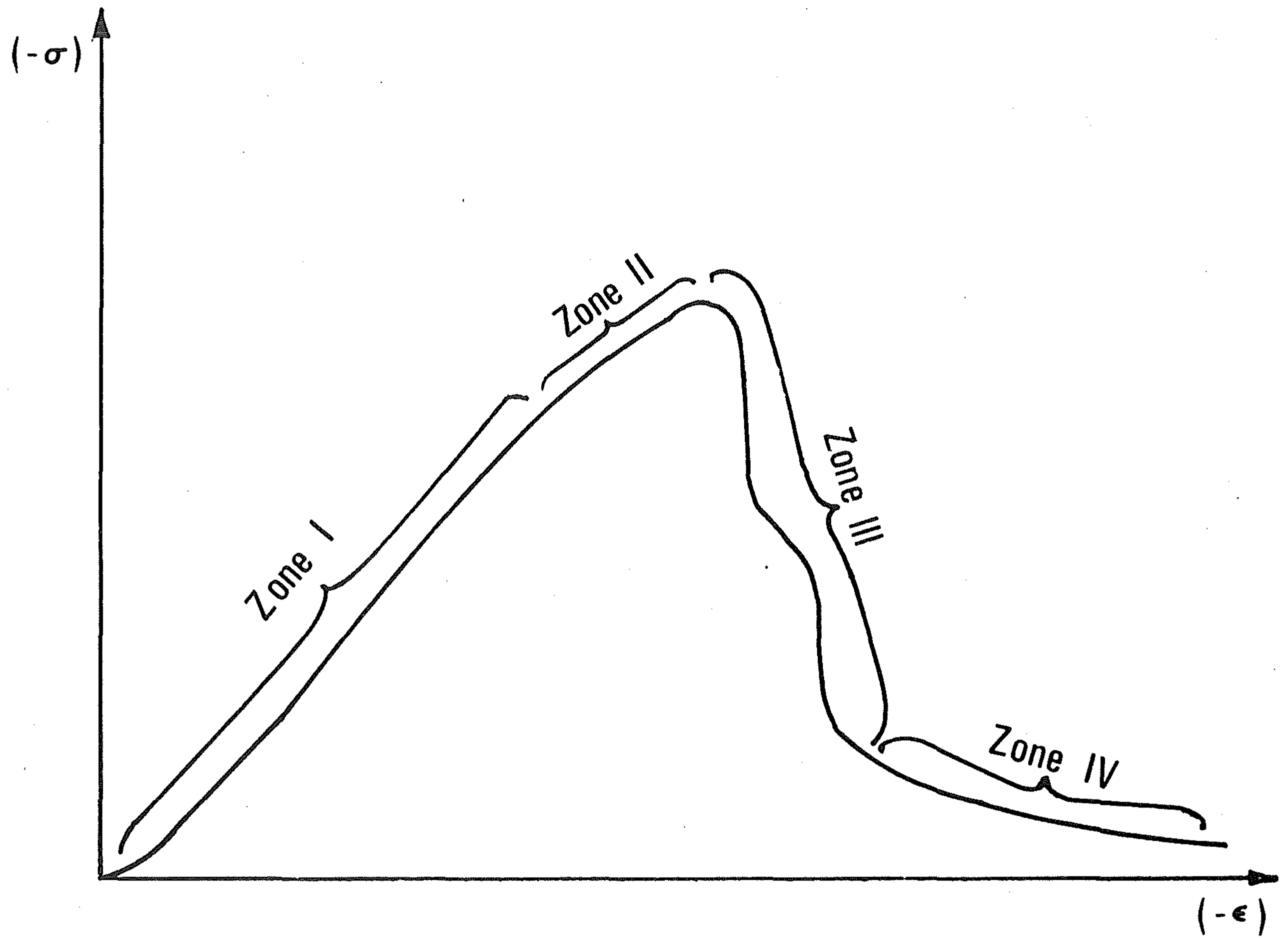

Figure 2 Typical Stress–strain Curve for Intact Rock.

volume increase even though the mean stress may be compressive.

Zone II is the region from the end of Zone I to maximum strength where, for relatively low mean pressures, microscopic cracks propagate and coalesce. For large confining pressures the deformation experienced in this region may be much larger and more ductile in nature (Bieniawski-67_b). It is a rate sensitive region; the slower the strain rate the lower the ultimate stress capacity. In this zone considerable damage is done to the material (often concentrated in a damage zone surrounding an advancing major crack (Hoagland-73)) of an anisotropic nature. Thus any unloading from this region generally yields a large hysteresis loop and upon subsequent reloading the response is anisotropic in nature.

The third behavioral zone covers a post peak strength region characterised by unstable crack propagation and a rapidly descending stress-strain curve. For Zone III the type of testing machine and the sample shape has a significant effect upon the measured results of laboratory tests (Brace-64, Bieniawski-67_b, Kupfer-69, Hoagland-73). Thus there is no general agreement as to the precise nature of the response in this zone (contrast Bieniawski-67_b and Wawersik-70). The authors tend to favor the interpretation of Wawersik-70, i.e., that the behavior in this zone is for many rocks catastrophic.

The fourth zone is a residual strength region for compressive stress states; the quantitative laboratory results for this zone may not be meaningful due to the presence of end effects, etc. However, this zone has a physical basis since even a pile of rubble has residual strength for certain stress states (Gardner-69).

For non-proportionate loadings the above described zones may lose their identity and be of little value in describing the behavior of rock. The loss of orderliness, as caused by more complicated stress and strain histories, in the response of rocks once they have entered Zones II, III or IV may be appreciated by consideration of the following example. Consider the several possible subsequent behaviors of a rock which has experienced a tensile fracture due to a tensile strain in the x direction and contrast the different responses to:

(a) a further increase in the tensile strain in the x direction,

(b) a reversal of strain so as to close the crack and then yield a compressive strain in the x direction, and

(c) loading of the specimen in the y direction, etc.

B. DEVELOPMENT OF MATHEMATICAL MODEL TO CHARACTERIZE THE STRUCTURAL BEHAVIOR OF ROCK

1. Representative Volume:

The characterization of the mechanical properties of composite materials has traditionally been based upon the concept of the "representative volume" (e.g., see Herrmann-63_b). The representative volume is a hypothetical element of idealized material whose structural properties are an approximation to the average properties of the material under consideration; the averaging process is extended over a volume commensurate in size with the level of structural interest. For structural analysis purposes the representative volume may be treated as if it were homogeneous. For example, if the structural behavior of steel is of interest, the properties of the appropriate representative volume would be equal to the microscopic properties of the steel averaged over a sufficiently large volume so that the effects of individual grains and grain boundaries would not be apparent. At the structural level of observation steel is therefore considered to be microscopically homogeneous even though its crystaline nature is recognized (it could, of course be macroscopically inhomogeneous due to changes in heat treatment or composition).

For structural analysis purposes the representative volume for a rock mass should represent the properties on a scale commensurate with the dimensions of the smallest structural feature that must be considered in detail. Thus, the scale of observation will be quite different for the purposes of a detailed theoretical study of the end effects in a small laboratory test specimen from that necessary for the structural analysis of a rock structure which may be hundreds of feet in extent, and for which the dimensions of the smallest finite element used in the analysis may be tens of feet. It is the objective of this study to initiate the development of a characterization for the structural properties of rock masses, in terms of a representative volume, that may be used in the analysis of rock structures of varying sizes. Its use for applications with greatly differing dimensions, will of course, require different values for the several parameters which describe the model, e.g., degree of variability, etc.

In order to be able to concentrate upon the main objective of the current study, i.e., formulation of the basic procedures for the development of a representative volume for rock masses, two restrictions were placed upon the scope of the study. Firstly, the development was limited to the case of plane strain. The condition of plane strain was selected because even though it is considerably easier to model than general three-dimensions, it still retains all the salient features of the more general case, and, in addition, it is an approximation that is often employed in the analysis of rock structures. Secondly, certain effects were neglected either because at this point they would greatly complicate the development (e.g., rate effects) and/or because for in situ conditions they are of secondary importance (e.g., the effect of the closure of microscopic cracks on the behavior of rock for near stress free conditions).

Representative volumes are, in general, composed of several different materials or phases (e.g., for steel it consists of grains with an assortment of orientations, and grain boundaries). The spatial (geometric) arrangement of these several constituents may be deterministic (as is the case for a matrix enclosing carefully placed reinforcing elements) or non-deterministic (as is the case for the grains and grain boundaries of steel). When the spatial relationship of the several different phases is non-deterministic they are usually visualized as being coexistent, i.e., in some sense they are considered to each occupy the same space (this approximation may lead to certain undesirable effects, and its modification will be the subject of a future study).*

For real materials the interaction of the several constituents is such that equilibrium and compatibility (with proper allowance being made for possible cracking) are satisfied. For non-deterministic models of materials it is generally impossible to simultaneously satisfy both these conditions. Thus it is common practice to develop an approximate characterization based upon the satisfaction of one of these two conditions and an approximation to the other (Hashin-64). It has been shown by Paul (60) that utilizing one or the other of these two extremes (i.e., satisfying compatibility and approximating equilibrium or vice versa) leads to bounds for the properties of the representative volume. However, because of the additional approximations used in the formulation of a non-deterministic representative volume, they are not necessarily bounds for the properties of the actual material. In addition, even if they were truly bounds of the actual properties, their use in the analysis of a complicated structure does not in general readily lead to bounding statements concerning the results (e.g., stress and strain predictions) of the analysis. Representative volumes made up of several different strengths of coexistent constituents, which are required to satisfy equilibrium yield the prediction that the model reaches its maximum stress capacity when the weakest constituent reaches incipient failure. Because this is obviously not true for most composite materials equilibrium models are seldom used.

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* In a sense this approximation is not completely adhered to in the current study, i.e., cracks can be considered as a separate phase, and in this study they are not totally considered as coexistent with the other phases.

For this study the model will be based upon compatibility considerations. However, because there is some question (Herrmann-67) concerning the desirability of utilizing properties which represent an extreme (i.e., bounds), a future extension of this study will include an assessment of the feasibility and desirability of basing the model upon a partial satisfaction of both compatibility and equilibrium.

Because of the difficulty of representing coexistent phases pictorially it may be of value, for illustrative purposes, to visualize the compatible plane strain model as a series of parallel plates (a corresponding visualization for three-dimensions is not possible). Each plate represents a phase and the thickness of each plate is in proportion to its representation in the total volume. Compatibility requires that each plate be subjected to a strain state identical to that of the representative volume (including thickness strain of zero; the simplistic visualization is misleading at this point). Because of the different response characteristics of the several constituents, the resulting stresses are not the same (i.e., equilibrium is only approximated). The stress state of the representative volume is the average of the stresses of the constituents (see pictorial representation in Figure 3).

The representative volume for a rock mass in a state of plane strain is considered to consist of I+1 particles.* The role of the I+1 particles is to simulate the variability of the rock mass. One

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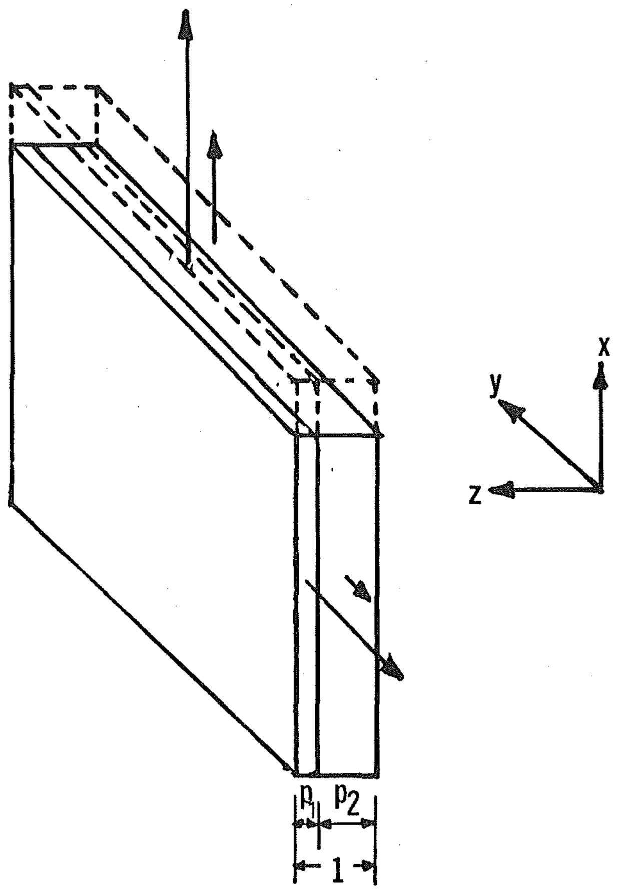
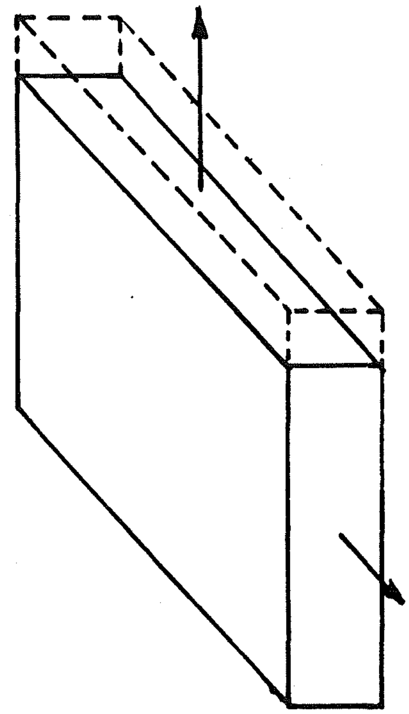
* As will be seen later, in a sense each particle may itself consist of several phases, i.e., intact rock and several planes of weakness or fracture.



Compatible Response of Constituents



Response of Representative Volume

Figure 3 Pictorial Representation of Compatible Representative Volume for a Two Phase Material in Plane Strain ($\epsilon_z = 0$) for Case of $\epsilon_y = 0, \epsilon_x \neq 0$.

of the particles (the I+1) accounts for the rubble phase. The remaining I particles are considered to be rock initially intact with the exception that they may contain up to three systems of planes of weakness or fracture, of varying degrees of structural integrity (the concept of structural integrity is defined later). The relative proportion (P_i) of each of the I+1 particles must be initially specified. These proportions may be fixed or be permitted to change during the course of the representative volume's stress and strain history. If this option is utilized, as the I intact particles experience damage, a certain portion of each is irreversibly assigned to the rubble phase. This option is included to account for the fragmentation which may accompany cracking (Handin-63, Brace-64).

Each of the I+1 particles is required to experience the same strain state as the representative volume (as was noted before the desirability of this assumption is open to question and will be the subject of a future inquiry)*, i.e.,

$$[\Delta\varepsilon]_i = [\Delta\varepsilon] \qquad i = 1 \rightarrow I+1 \qquad (3)$$

$[\Delta\varepsilon]_i$ = strain state of particle i

$[\Delta\varepsilon]$ = strain state of the representative volume ($\Delta\varepsilon_{x_N}$, $\Delta\varepsilon_{y_N}$, $\Delta\gamma_{xy_N}$)

also $\Delta\varepsilon_{xz_i} = \Delta\varepsilon_{yz_i} = \Delta\varepsilon_{z_i} = \Delta\varepsilon_{xz} = \Delta\varepsilon_{yz} = \Delta\varepsilon_z = 0$ (plane strain)

The stress state for the representative volume is given by the

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* In the following, for the sake of simplicity, the subscript N referring to the increment number is not displayed.

following summation:

$$[\Delta\sigma] = \sum_{i=1}^{I+1} P_i \, [\Delta\sigma]_i \tag{4}$$

Where $[\Delta\sigma] = (\Delta\sigma_x, \Delta\sigma_y, \Delta\tau_{xy})$, etc.

Note $\sum_{i=1}^{I+1} P_i = 1$

For each particle the relationship between the incremental stress and strain is written as (the increment number N is implied):

$$[\Delta\sigma]_i = [C]_i[\Delta\varepsilon]_i + [L]_i \tag{5}$$

Using equs. (3) and (4)

$$[\Delta\sigma] = \sum_{i=1}^{I+1} P_i \, \{ [C]_i[\Delta\varepsilon] + [L]_i \}$$

or

$$[\Delta\sigma] = \{ \sum_{i=1}^{I+1} P_i \, [C]_i \} [\Delta\varepsilon] + \sum_{i=1}^{I+1} P_i \, [L]_i$$

Comparing this expression with equ. (1) yields:

$$[C] = \sum_{i=1}^{I+1} P_i \, [C]_i \tag{6}$$

$$[L] = \sum_{i=1}^{I+1} P_i \, [L]_i \tag{7}$$

Each of the I intact particles is considered to be the accumulation of rock, of like structural integrity and distributed (in a non-deterministic manner) throughout the representative volume. For each of these particles an integrity factor F_i is assigned. The strength parameters for each particle (e.g., tensile strength σ_{T_i}) are then written as the product of F_i and the average value of the corresponding parameter for the material as a whole, i.e.,

$$\sigma_{T_i} = F_i \, \sigma_T \qquad i = 1 \rightarrow I \tag{8}$$

Thus, for each particle complete correlation is assumed between the several strength parameters; this idealization is based upon the assumption that they are all manifestations of one fundamental property (herein called structural integrity). The modification of this assumption would require knowledge of the actual correlations of these parameters; experimental information concerning such correlations appears to be scant.

Figure 4 gives an example of a histogram of structural integrity developed from the variability of the compressive strength of Inada granite given by Yamaguchi-70. The degree of scatter of the structural

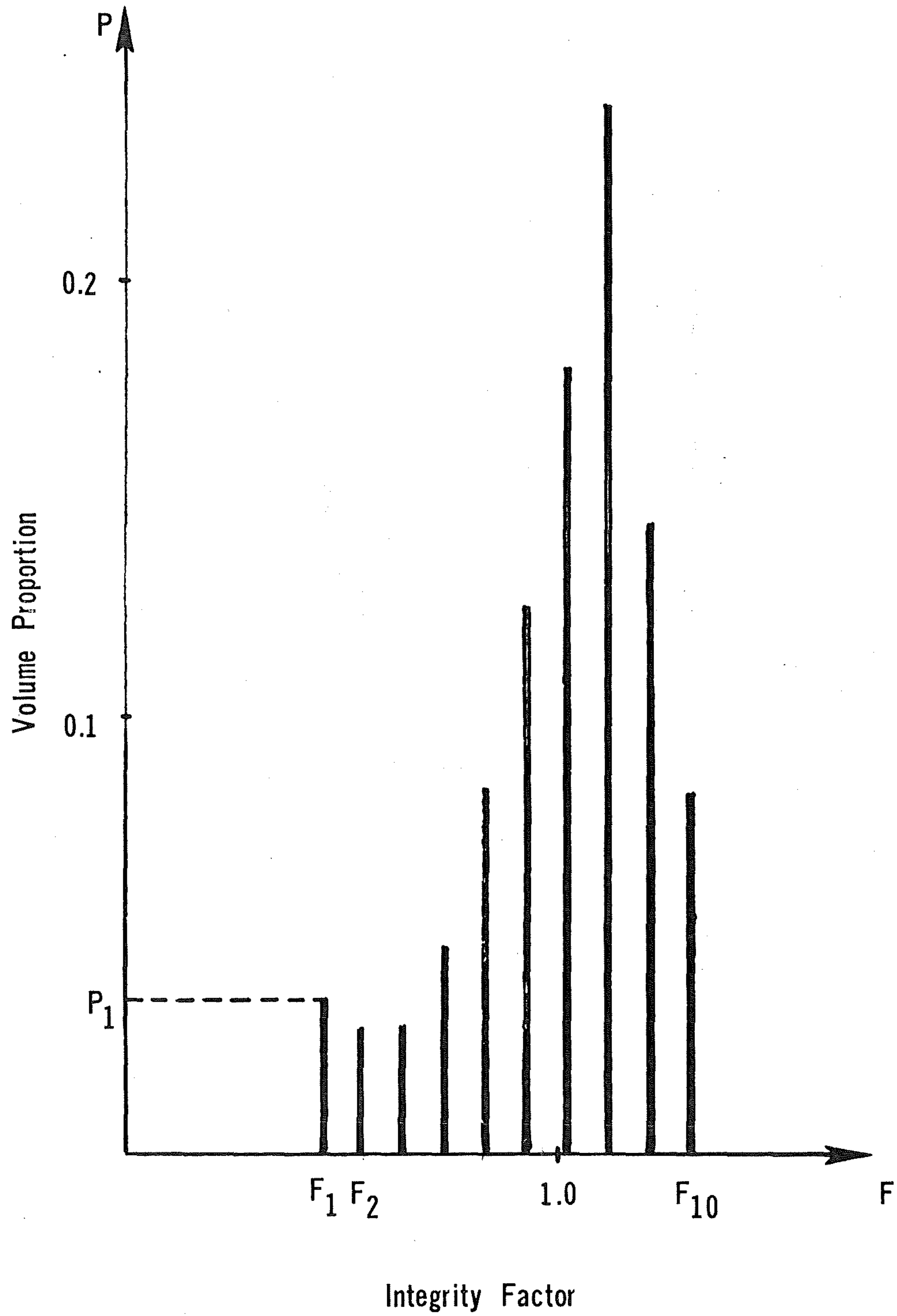

Figure 4 Example Histogram of Structural Integrity.

integrity, of course, depends upon the size of the representative volume (i.e., size effect, see Evans-58, Brady-70, Hoagland-73) or, more precisely, upon the size of the finite elements to which the representative volume properties are being applied. Rigorous accounting for size effects would require knowledge of the correlation of the integrity factor from location to location in the rock (a quantity about which very little is known) and would lead to the prediction of non-deterministic structural properties. In general finite element and other structural analysis methods are not, at this time, sufficiently well advanced to accept such probabilistic descriptions of structural properties (some exceptions may be found in the works by Langland-71 and D'Andrea-74). Thus at present size effects are ignored and therefore the histogram for structural integrity should represent, as nearly as possible, the variability of strength properties on a scale commensurate with the level of observation for which the structural properties are to be used.

The stiffness properties (e.g., initial modulus) are much less variable than the strength properties, and thus, for the purpose of this study are taken to be the same for all particles. If, however, information concerning the variability of the stiffness properties should become available, and assuming perfect correlation with strength properties, it could be incorporated into the model with ease.

At this point the structural behavior of the individual phases is considered:

2. Rubble Phase:

Because the main interest of this study is the characterization of relatively sound rock the rubble phase is considered to be of minor importance and its behavior is grossly simplified. Should it become desirable in the future to consider exceptionally unsound rock, even to the extent that it approaches a granular mass in composition, then this approximation would need to be revised.

Because the shear resistance of the rubble is small compared to that of the intact rock it is approximated as zero; thus, it is assumed that the rubble has resistance only to compressive mean pressures*, i.e.,

$$[\Delta\sigma]_{I+1} = [C]_R\,[\Delta\varepsilon] + [L]_R$$

Where:

$$[C]_R = \begin{bmatrix} \overline{K}/3 & \overline{K}/3 & 0 \\ \overline{K}/3 & \overline{K}/3 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ and } [L]_R = [0] \qquad (9)$$

$$\text{Where } \overline{K} \equiv \begin{Bmatrix} K \text{ (bulk modulus)} \\ 0 \end{Bmatrix} \text{ for } \begin{Bmatrix} \varepsilon_x + \varepsilon_y < 0 \\ \varepsilon_x + \varepsilon_y \geq 0 \end{Bmatrix}$$

If the material is initially anisotropic the concept of bulk modulus has been appropriately generalized by Herrmann-72. For

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* Note: Throughout this report the "structural mechanics" sign convention is adopted, i.e., compressive strains are negative, etc.

simplicity, however, the following approximation is used (the subscript zero refers to the initial properties of the intact rock):

$$\overline{K} = 1/6\ (C_{11} + C_{22} + 4C_{12})_0$$

Because of the relative unimportance of this phase a constant bulk modulus is used for all compressive mean pressures even though it is suspected that it is a strong nonlinear function of the mean pressure (e.g., see Bridgemen-52).

3. Intact Phase:

The intact phase is utilized to represent the majority of the rock. As was noted in a previous section, the I particles of intact rock are assigned different values of the integrity factor, F, in order to simulate the varying strengths of rock contained within the representative volume. The different strengths are due to varying degrees of prior damage and/or variability introduced at the time of formation.

The several concepts used in the description of this phase are now discussed.

Orthotropic elastic response of sound rock: All rock within the intact phase which has not fractured is assumed to behave as a linear elastic material, i.e.,*

$$[\Delta\sigma] = [C]_e\ [\Delta\varepsilon]' + [L]_e \qquad (10)$$

* For the sake of simplicity the subscript i denoting the particle number and the subscript N denoting the increment number are not always explicitly noted in this section.

Where $[\Delta\sigma]$ = increment of stress for particle i

$[\Delta\varepsilon]'$ = increment of <u>elastic</u> strain

$[C]_e$ = elastic properties for the sound rock

$[L]_e$ = 0

For an isotropic material (E and ν are Young's modulus and Poisson's ratio, respectively):

$$[C]_e = \begin{bmatrix} \lambda + 2\mu & \lambda & 0 \\ \lambda & \lambda + 2\mu & 0 \\ 0 & 0 & \mu \end{bmatrix}$$

where

$$\mu = \frac{E}{2(1+\nu)}$$

$$\lambda = \frac{2\mu\nu}{1-2\nu}$$

Because it is felt that the nonlinearities (predominantly an elastic effect) experienced for very small stress levels, e.g., the beginning of zone I of Figure 2, contribute little to the behavior of in situ rock, they have been ignored (see discussion in Section III-A-2). If in the future it should prove to be desirable to include this phenomenon, it is felt that this could be accomplished by expressing the bulk modulus as a nonlinear function of the mean pressure (Herrmann-63_a).

The nonlinearity of zone II (Figure 2) is basically an inelastic phenomenon which is a consequence of the local fracturing process considered in the following paragraphs.

Transformation of stress and strain components: A preliminary need for the following developments is the transformation relating the stress and strain components in the x-y coordinate system (i.e., $[\Delta\varepsilon]$ and $[\Delta\sigma]$) to the components in an arbitrary n-s system, defined by the angle θ (i.e., $[\Delta\varepsilon]_\theta$ and $[\Delta\sigma]_\theta$, see Figure 5)*, i.e.,

$$[\Delta\sigma] = [T]\,[\Delta\sigma]_\theta \tag{11}$$

$$[\Delta\varepsilon]_\theta = [T]^T\,[\Delta\varepsilon] \tag{12}$$

where

$$[T] = \begin{bmatrix} \frac{1}{2}(1+\cos 2\theta) & \frac{1}{2}(1-\cos 2\theta) & -\sin 2\theta \\ \frac{1}{2}(1-\cos 2\theta) & \frac{1}{2}(1+\cos 2\theta) & \sin 2\theta \\ \frac{1}{2}\sin 2\theta & -\frac{1}{2}\sin 2\theta & \cos 2\theta \end{bmatrix} \tag{13}$$

The inverse transformations, i.e., between $[\Delta\sigma]$ and $[\Delta\sigma]_\theta$, etc., are obtained by replacing θ by its negative.

Representation of structural cracks: In the following paragraphs the term fracture plane is used to denote a system of approximately parallel macroscopic (sub-structural) planes of fracture. (Under certain circumstances such a system could conceivably contain only a single plane.)

In contrast, structural cracks are defined as major cracks

* The reason for employing this somewhat unorthodox notation for the stress components shown in Figure 5 will be apparent later.

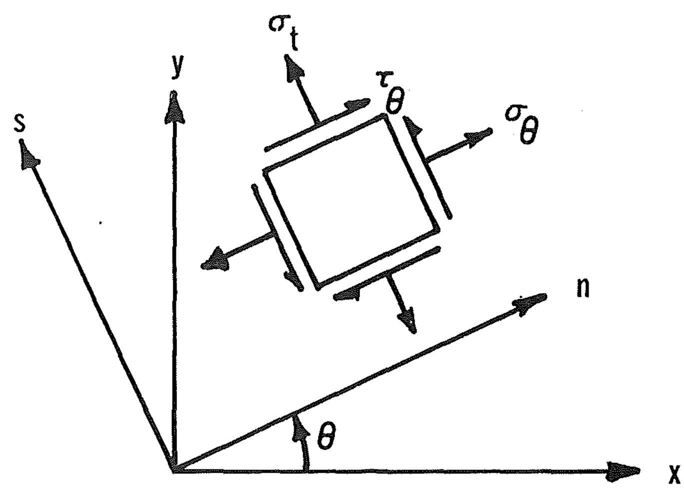

Figure 5 Stress Components in the n–s Coordinate System.

whose extent and spacing are larger than the minimum dimensions of the finite elements being used in the analysis of the structure. The treatment of structural cracks may vary depending on whether or not the crack exists prior to the beginning of the period of the analysis. Those structural cracks which exist prior to the beginning of the analysis, and whose spatial locations are well defined, may be modeled by joint elements (Goodman-68)*. A structural crack which develops during the course of the analysis can be accounted for in one of two ways: Either an attempt can be made to predict its actual course through the rock by introducing discontinuities in the finite element grid (e.g., see Pian-71 and Taylor-72) or a zone of fractured elements (i.e., elements with extensive internal damage) can be permitted to develop along the approximate path of the crack (a somewhat analogous procedure was used by Hossain-74). The characterization developed herein is sufficiently general to accommodate the latter form of failure prediction and in the authors' opinions, the second procedure is generally preferable because of the exorbitant computational cost involved in attempts to trace the actual paths of advancing cracks by means of discontinuities in the finite element grid.

Macroscopic cracks: In the following paragraphs the development of

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* It is to be noted that a finite element aligned with a crack, and having properties supplied by means of the characterization developed herein, may in fact be used as a joint element (thus a program using this characterization would have no need for a special joint element). The representative volume for such elements would have one natural plane of weakness with the appropriate orientation and stiffness properties of the joint.

macroscopic fracture planes in individual intact particles is considered in detail. The development of fracture planes in one or more of the I intact particles, of course, does not mean that the representative volume has failed; the representative volume does not have a complete fracture zone until all the intact particles have developed such planes. The appearance of fracture planes in some of the intact particles indicates that the material is experiencing inelastic behavior (e.g., zone II or III of Figure 2). If loading is reversed after only a portion of the I particles have developed fracture planes, then the representative volume will exhibit anisotropic behavior as part of its future response characteristics (careful tests by Wawersik-70 demonstrated such a phenomenon).

Each of the I intact particles is permitted to contain up to three systems of fracture planes. These systems consist of a combination of (zero to three) prescribed pre-existent planes (i.e., present prior to the period covered by the analysis) and planes developed during the course of the analysis. A maximum of three was selected for the following reasons: In general, it appears that highly fractured rock contains, at most, four systems of fracture planes (Pomeroy-71), one of which, most likely, is parallel to the plane of the two dimensions currently under consideration and thus ineffectual in plane strain. Secondly, it appears that for two-dimensions movements along three fracture planes is sufficient to relieve the stress to such a degree that, for all practical purposes, it is impossible to develop a fourth fracture plane.

Finally the computational cost involved in using the model is dependent upon the maximum permissible number of failure planes.

Because of the different types of discontinuities (fracture zones) which may occur in rock, e.g., a single well defined crack, a series of closely spaced fine cracks, a region of concentrated deformation caused by many unjoined cracks and/or plastic-like dislocations, the failure criterion used to predict the formation of such fractures must be quite general.

Numerous different failure criteria have been proposed for rock (e.g., see Handin-63, Murrell-65, Bieniawski-67_a, Brady-70, and Herget-70) none of which have proven to be entirely satisfactory (e.g., see Brace-64, Howe-73). The chief obstacles which have apparently prevented the firm establishment of a failure criterion for rock, are its variability, the numerous types of rock, the difficulty in achieving homogeneous strain and stress states for even simple test specimens, the lack of acceptable experimental procedures for achieving multi-axial stress and strain states in the laboratory, and finally the confusion that exists between the concepts of a macroscopic and a structural failure criterion (e.g., see Bieniawski-67_a); between a failure criterion and a sliding criterion; and between a yield and a failure criterion.

The failure criterion that is needed for the intact particles is a criterion for the formation of macroscopic fracture planes. (If one should choose to set I=1 then, depending upon the size of the representative volume, a gross structural failure criterion might be required.)

The two most widely used failure criteria for rock are the Mohr and the Griffith (or modifications thereof). While apparently sound experimental evidence has been presented for the justification of each, it appears to the authors that the Griffith criterion is somewhat more rational; however, its use does not appear to result in sufficiently improved predictions to justify, at this time, the greater complexities involved (e.g., see Brace-64). Thus, for this study, the Mohr criterion is used. While there appears to be some evidence that the envelope should not be straight (e.g., see Brace-64, Murrell-65), it is assumed to be linear in this report for simplicity. The straight portion is, however, terminated in the tension region by a tension failure "cut off" criterion. It is suggested that careful considerations be given to the use of a parabolic envelope in future work.

It needs to be emphasized that what is under consideration at this point is a failure initiation condition not a criterion for sliding once the fracture has occured. Under certain conditions (particularly for large hydrostatic pressures) such a clear distinction may not be justified, because it appears that a considerable amount of deformation takes place across the fracture zone before it is completely formed (e.g., see Handin-63). This phenomenon is discussed in more detail later.

The fracture criterion for particle i is written as:

$$|\tau_\theta|_i = c_{o_i} - f_{o_i}\,\sigma_{\theta_i} \quad \text{(shear fracture} \tag{14}$$

or $$\sigma_{\theta_i} = \sigma_{t_i} \quad \text{(tension failure-note } \sigma_{t_i} < c_{o_i}/f_{o_i}\text{)} \tag{15}$$

Where τ_{θ_i} and σ_{θ_i} are the shear and normal stress components on the fracture plane. For the sound portions of the rock the parameter c_{o_i} can be interpreted as the "internal" cohesion and f_{o_i} as the coefficient of "internal" friction, i.e., parameters which describe a failure criterion but do not necessarily have any physical significance (for a natural plane of weakness they are the cohesion and friction associated with the plane). If the rock is inherently orthotropic then it is to be expected that c_{o_i}, σ_{t_i} and f_{o_i} would be continuous functions of orientation (e.g., see Jaeger-60). These functions of θ describe the variation due to anisotropy of the sound rock, in addition, there may be discontinuous changes in their values due to the presence of natural planes of weakness (bedding planes, etc.); these special values are specified separately. Because f has a somewhat restricted range of values it is assumed that it is not a function of θ (this assumption could easily be revised). The orthotropic nature of c_{o_i} is written as (recall that F_i is the integrity factor):

$$c_{o_i}(\theta) = F_i'\{\tfrac{1}{2}(1 + \cos 2\theta)\, c_o(x) + \tfrac{1}{2}(1 - \cos 2\theta)\, c_o(y)\}$$

or

$$c_{o_i}(\theta) = F_i c_o(x)\, [\tfrac{1}{2}(1 + \cos 2\theta) + \tfrac{1}{2}(1 - \cos 2\theta)\, R] \qquad (16)$$

Where $R = \dfrac{c_o(y)}{c_o(x)}$

It is assumed that c and σ_t are both characteristics of the same fundamental strength property and thus have the same value of R (this assumption would be easy to revise), i.e.,

$$\sigma_{t_i}(\theta) = F_i \sigma_t(x) \left[\frac{1}{2}(1 + \cos 2\theta) + \frac{1}{2}(1 - \cos 2\theta) R \right] \qquad (17)$$

The fundamental question that must be addressed at each increment, for each of the I intact particles, is whether or not the particle has already developed its allowed maximum of three fracture planes, and if not, whether or not equ. (14) or (15) might be satisfied during the course of the increment. If the particle has not developed its three planes of fracture, two possibilities for the satisfaction of equ. (14) or (15) are considered:

First, prescribed planes of weakness which have not already failed are inspected. It is to be recalled that up to three naturally occuring planes of weakness or fracture may be prescribed, i.e., θ specified. The values of c_o, σ_t and f_o for each of these planes are of course different than the values for the sound rock.*

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* If the sum of the number of planes considered in this step, and the number of existing planes of fracture is equal to three, then the considerations described in the subsequent paragraphs are skipped; the reasons for this action is as follows: Because the magnitudes of c_0 and σ_t for the natural planes of weakness are expected to be considerably less than for the surrounding sound rock and because the natural planes tend to be mutually perpendicular to each other, it is anticipated that any non-fractured natural planes of weakness are approximately perpendicular to the existing fracture planes. This expectation, along with the condition that the natural weakness planes are substantially weaker than the surrounding rock, leads to the conclusion that if any additional fracturing should take place during this increment, that it would be along the non-fractured natural weakness planes and not in the sound rock.

A pre-existing plane of failure is prescribed as a plane of weakness with zero strength parameters - in the present analysis all such planes are considered to be initially closed.

Secondly, for the sound rock the orientation (defined by a critical value of θ) of a possible fracture plane which would most likely cause the satisfaction of equ. (14) or (15) is determined. Once this plane is determined, equs. (14) and (15) are checked for possible satisfaction. A critical orientation determined from this consideration is called a stress-induced failure orientation as opposed to the specified orientations of the natural occuring planes of weakness.

If these two considerations reveal more than one possible failure plane, failure is permitted to occur on the one which is the earlier to reach the critical state in the course of the increment (this failure would, of course, so alter the stress state that the second would not occur).

The determination of the possible stress induced failure plane orientation proceeds as follows:

Both equs. (14) and (15) are expressed in a common form, i.e.,

$$\eta \, \tau_\theta = c - f \, \sigma_\theta \qquad (18)$$

For shear failure:

$$\eta = \frac{\tau_\theta}{|\tau_\theta|} , \ c = c_{o_i} , \ f = f_{o_i}$$

For tension failure:

$$\eta = 0, \; c = \sigma_{t_i} f_{o_i}, \; f = f_{o_i}$$

Utilizing the inverse of equ. (11) the above expression is written in the form:

$$\eta \left\{ \frac{\sigma_y - \sigma_x}{2} \sin 2\theta + \tau_{xy} \cos 2\theta \right\} = c -$$

$$f \left\{ \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cos 2\theta + \tau_{xy} \sin 2\theta \right\} \qquad (19)$$

Expressing the above equation in matrix notation yields:

$$[Q]^T [\sigma] + c = 0 \qquad (20)$$

The vector Q is defined as:

$$[Q] = \begin{bmatrix} -\frac{1}{2}[f(1 + \cos 2\theta) - \eta \sin 2\theta] \\ -\frac{1}{2}[f(1 - \cos 2\theta) + \eta \sin 2\theta] \\ -[f \sin 2\theta + \eta \cos 2\theta] \end{bmatrix} \qquad (21)$$

The questions that must now be answered are, will this equation be satisfied during the course of increment N, and if so, at what point in the increment and for what critical value of θ. Denote the stress state at the beginning of the increment as $[\sigma]_{N-1}$ and the apparent incremental change as $[\Delta\sigma]_N$. Assume that during the course of the increment up to the point of the formation of the fracture plane, the changes in the several stress components are proportional (this assumption is one of the limiting factors on the permissible size of the increments), i.e.,

$$[\sigma] = [\sigma]_{N-1} + k\,[\Delta\sigma]_N \qquad 0 \leq k \leq 1 \tag{22}$$

Substituting equ. (22) into equ. (20) yields:

$$u(\theta) - k\,v(\theta) = 0 \tag{23}$$

where

$$u(\theta) = c + [Q]^T[\sigma]_{N-1} \tag{24}$$

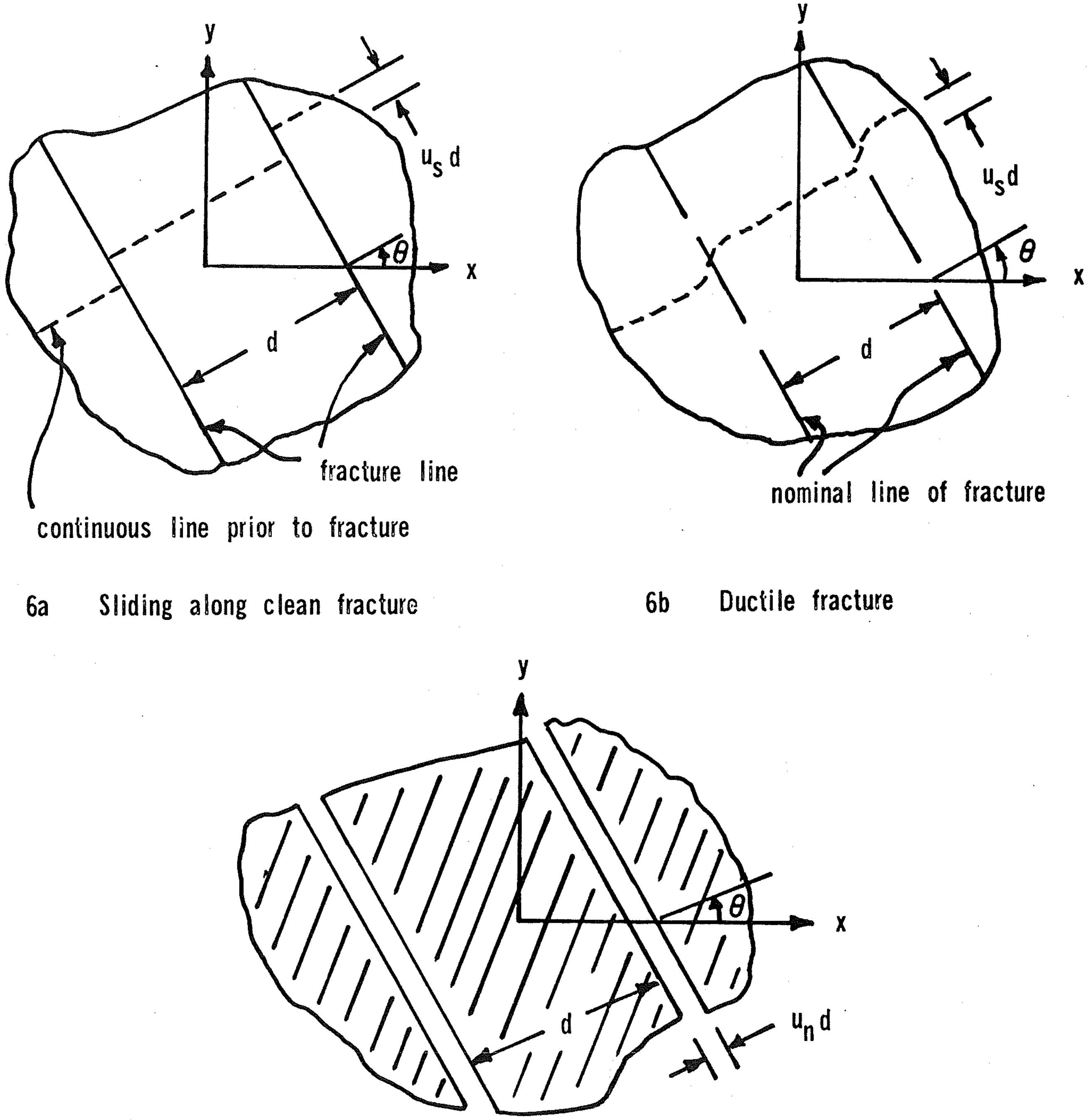
$$v(\theta) = -\,[Q]^T[\Delta\sigma]_N \tag{25}$$

Thus:

$$k = \frac{u(\theta)}{v(\theta)} \tag{26}$$

What must now be determined is the value of θ that yields the minimum positive value of k (the definition of k, equ. (22), limits its range to $k \geq 0$). The determination of the critical value of θ and the corresponding value of k is quite involved; the details of these calculations are given in Appendix A.

Deformation resulting from movement along fracture planes: The relative movements of the opposing faces of a fracture plane may be expressed in terms of sliding and separation (i.e., opening of the crack). Sliding deformation of a fracture plane is illustrated for a clean fracture in Figure 6a and a ductile type fracture zone in Figure 6b. It will be seen later that it is not necessary to



6a Sliding along clean fracture

6b Ductile fracture

6c Open fracture

Figure 6 Deformation associated with fracture distinguish between these two types of sliding deformations. Opening of a fracture is illustrated in Figure 6c. The effect of this deformation upon an element of rock oriented with the n-s axes is illustrated in Figure 7, in addition, the "equivalent distributed deformation" of such an element is also shown; this deformation is given by the following expressions (the primes denote strain due to movement along a fracture zone)*:

$$\Delta\varepsilon_n'' = \frac{u_n d}{d} = u_n$$

$$\Delta\varepsilon_s'' = 0$$

$$\Delta\gamma_{ns}'' = \frac{u_s d}{d} = u_s$$

Using the inverse of equ. (12) to transform these strains to the x-y system yields:

$$[\Delta\varepsilon]'' = u_n\,[A] + u_s\,[B] \qquad (27)$$

where

$$[A] = \begin{bmatrix} \frac{1}{2}(1 + \cos 2\theta) \\ \frac{1}{2}(1 - \cos 2\theta) \\ \sin 2\theta \end{bmatrix}$$

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* Note that u_n and u_s are not actual displacement quantities but rather displacements per unit of distance between cracks (i.e., strain quantities); the actual opening of a fracture is u_n d, etc.

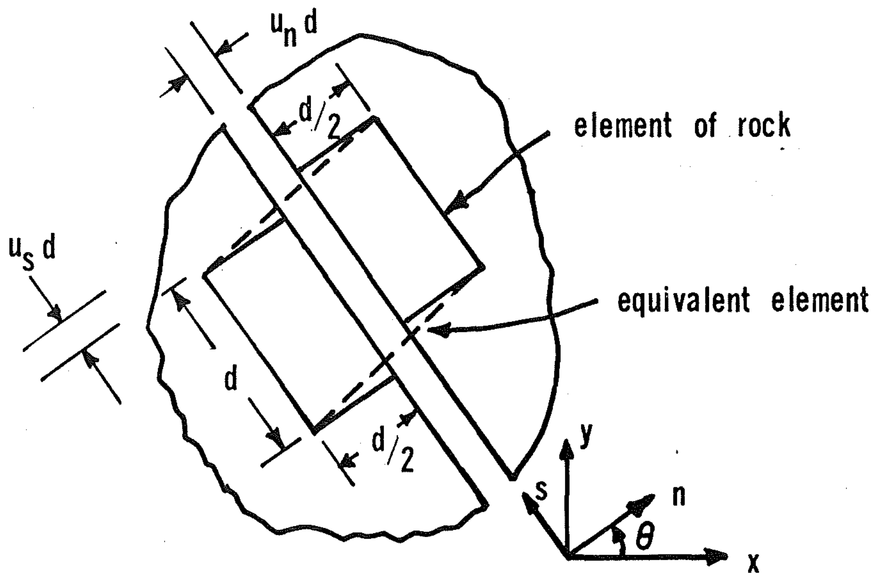

Figure 7 Deformation of Rock Element Caused by Movement along Fracture.

$$[B] = \begin{bmatrix} -\frac{1}{2} \sin 2\theta \\ \frac{1}{2} \sin 2\theta \\ \cos 2\theta \end{bmatrix}$$

Note (see equ. (21)) that $[Q] = - f[A] - \eta[B]$.

Thus, movement along a fracture zone results in deformation (i.e., strain) of the representative volume. Such strain is, of course, not uniformly distributed, as is usually visualized to be the case for elastic strain, but instead concentrated at the fracture zones (this concentration of strain is somewhat akin to the discontinuous nature of the plastic strain in metals that develops at dislocations).

Mechanics of fractured particles: Procedures must now be established for determining whether or not sliding (u_s) and/or separation (u_n) occurs along a fractured zone and their respective magnitudes. In addition, the state of stress at such a fracture plane must be investigated.

A little reflection indicates that a particle, having as many as three fracture planes, may experience several different modes of structural response, i.e., no movement along fractures, sliding along one fracture, sliding along one fracture and opening of a second fracture, etc. In order to simplify the logic of the model it is important to eliminate from consideration those response modes which are either impossible or highly unlikely or are expressable in terms of other modes*. Within each increment, for a given particle, only a single transition from one type of behavior to another is permitted; symbolically this is illustrated in Figure 8. These two divisions of the increment are called "intervals". If the incremental strain changes are relatively small, it appears that the most likely combinations of behaviors for a given particle are limited to those listed in Table 1 (in all cases b may range from 0.0 to 1.0). While it is unlikely, for small strain increments, that any other response mode will occur, if one should (e.g., closure of two fracture planes during the same increment) a small error will be introduced. However, this error may be corrected in the next increment (see Appendix A). The logic of selecting, for a given increment, the proper response mode is discussed in a following section.

The precise definition of the factor b, used in Figure 8, is such that the strains for the two response regions are:

$$[\Delta\varepsilon]_a = (1 - b)\ [\Delta\varepsilon]_N$$

$$[\Delta\varepsilon]_b = b\ [\Delta\varepsilon]_N$$

For particle i in response region a, write the incremental stress-strain relationship in the form:

$$[\Delta\sigma]_{i_a} = [C]_{i_a}\ [\Delta\varepsilon]_a + [L]_{i_a}$$

---

* For example, the behavior of a particle with three open fractures can, by the introduction of rigid body motion (accompanied by no change in the strain or stress states), be expressed in terms of the behavior of one with two open cracks.

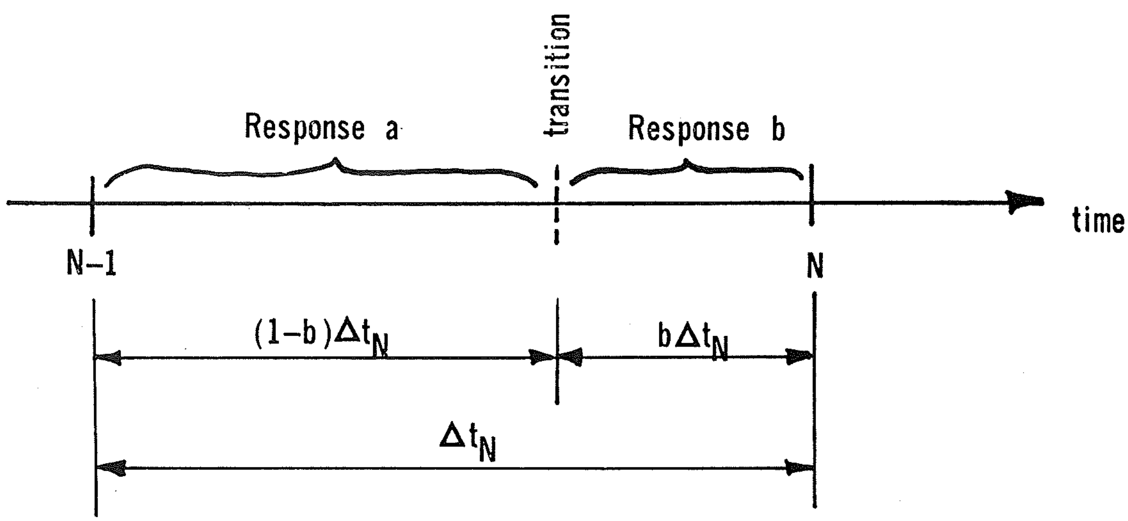

Figure 8 Symbolic Representation of the Transition in Response Mode Permitted in a Given Increment.

Response a	Nature of Transition	Response b
Elastic response of sound rock	New shear fracture or renewed activity of existing closed fracture	Combination elastic response and sliding along fracture
Elastic response of sound rock	New tension fracture or incipient opening of existing fracture	Elastic response of rock with one open crack
Elastic response of rock with one open crack	New shear fracture or renewed activity along existing closed fracture	Combination elastic response (one open crack) and sliding along a closed crack
Elastic response of rock with one open crack	Formation of another tension fracture or incipient opening of existing closed crack	Elastic response with two open cracks
Elastic response of rock with one open crack	Closure of crack	Elastic response of rock without movement along closed crack
Elastic response of rock with two open cracks	Closure of one crack	Elastic response with one open crack without movement along closed crack

Table 1 - Permissible Combinations of Response for a Particle in a Given Increment.

Use has been made of equ. (3); the determination of $[C]_{i_a}$, etc., is discussed in subsequent sections. Noting the definition of b the above expression is written in the form:

$$[\Delta\sigma]_{i_a} = [C]_{i_a} (1 - b) [\Delta\varepsilon]_N + [L]_{i_a}$$

likewise

$$[\Delta\sigma]_{i_b} = [C]_{i_b} b[\Delta\varepsilon]_N + [L]_{i_b}$$

The total stress increment is the sum of these two expressions, i.e.,

$$[\Delta\sigma]_i = [\Delta\sigma]_{i_a} + [\Delta\sigma]_{i_b}$$

Thus:

$$[\Delta\sigma]_i = \{ (1-b)[C]_{i_a} + b[C]_{i_b} \} [\Delta\varepsilon]_N + [L]_{i_a} + [L]_{i_b}$$

Comparing the above expression with equ. (5) yields:

$$[C]_i = (1-b)[C]_{i_a} + b[C]_{i_b} \qquad (28)$$

$$[L]_i = [L]_{i_a} + [L]_{i_b} \qquad (29)$$

The appropriate expressions for [C] and [L] must now be established for each of the possible response mechanisms (including the effects of the transition to the mechanism, i.e., fracturing and the opening or closing of the resulting cracks). In formulating these expressions two approximations are made. A closed joint has infinite normal stiffness and a sliding closed joint experiences no normal deformation. There is good experimental evidence to refute both of these assumptions (see discussion in Section III-A-1). Thus an obvious future improvement to the model will be to incorporate a more realistic description of joint behavior such as that suggested by Goodman-72.

Response of sound rock: For situations when there is no movement along any fracture planes the particle is considered to behave elastically, i.e., obeys equ. (10).

Response with one open crack: Consider a particle with an open fracture; if this fracture were not previously open it is either newly formed at the beginning of the interval or the result of the opening of a previously closed fracture, see Table 1. If the fracture were open at the beginning of the interval it may remain open or it may close at the end of the interval.

The total strain $[\Delta\varepsilon]$ (the subscript i indicating the particle number and a or b indicating the interval, see Figure 7, are not displayed for simplicity) consists of an elastic strain $[\Delta\varepsilon]'$ and a portion due to the relative movement of the faces of the open crack $[\Delta\varepsilon]''$ (equ. (27)), i.e.,

$$[\Delta\varepsilon] = \Delta u_n \, [A] + \Delta u_s \, [B] + [\Delta\varepsilon]' \qquad (30)$$

The stress is related to the elastic strain by equ. (10), i.e.,

$$[\Delta\sigma] = [C]_e \, [\Delta\varepsilon]'$$

Solving for $[\Delta\varepsilon]'$ from equ. (30) and substituting into the above expression yields:

$$[\Delta\sigma] = [C]_e \{ [\Delta\varepsilon] - \Delta u_n [A] - \Delta u_s [B] \}$$

Define:

$$[E] = [C]_e [A] , \qquad [D] = [C]_e [B] \tag{31}$$

Thus:

$$[\Delta\sigma] = [C]_e [\Delta\varepsilon] - \Delta u_n [E] - \Delta u_s [D] \tag{32}$$

Using the definitions of the [A] and [B] matrices, the expressions (inverse of equ. (11)) for the normal ($\Delta\sigma_\theta$) and the shear ($\Delta\tau_\theta$) components of stress acting across the fracture are written in the form:

$$\Delta\sigma_\theta = [A]^T[\Delta\sigma] \tag{33}$$

$$\Delta\tau_\theta = [B]^T[\Delta\sigma] \tag{34}$$

For a fracture continually open the values of $\Delta\sigma_\theta$ and $\Delta\tau_\theta$ are zero; for a newly formed fracture they are equal to the <u>negatives</u> of the stresses existing across the plane at the time of fracture. Defining σ_{θ_o} and τ_{θ_o} to have the appropriate values and using equs. (32), (33) and (34) yields:

$$\Delta\sigma_{\theta_o} = [A]^T \{ [C]_e[\Delta\varepsilon] - \Delta u_n [E] - \Delta u_s [D] \} \tag{35}$$

$$\Delta\tau_{\theta_o} = [B]^T \{ [C]_e[\Delta\varepsilon] - \Delta u_n [E] - \Delta u_s [D] \} \tag{36}$$

Define (note: $[A]^T[D] = [B]^T[E]$):

$$[\chi] = \begin{bmatrix} [A]^T[E] & [A]^T[D] \\ \text{Symm.} & [B]^T[D] \end{bmatrix}$$

Using equ. (31) and the fact that $[C]_e = [C]_e^T$, equs. (35) and (36) may be written in the form:

$$[\chi] \begin{bmatrix} \Delta u_N \\ \Delta u_s \end{bmatrix} = \begin{bmatrix} [E]^T[\Delta\varepsilon] - \Delta\sigma_{\theta_o} \\ [D]^T[\Delta\varepsilon] - \Delta\tau_{\theta_o} \end{bmatrix}$$

The solution of these two simultaneous equations yields values for the increments of deformation of the open fracture plane.

$$\Delta u_n = [H]^T[\Delta\varepsilon] + \psi_{11}\,\Delta\sigma_{\theta_o} + \psi_{12}\,\Delta\tau_{\theta_o} \qquad (37)$$

$$\Delta u_s = [M]^T[\Delta\varepsilon] + \psi_{12}\,\Delta\sigma_{\theta_o} + \psi_{22}\,\Delta\tau_{\theta_o} \qquad (38)$$

where

$$[H]^T = -\{\psi_{11}\,[E]^T + \psi_{12}\,[D]^T\} \qquad (39)$$

$$[M]^T = -\{\psi_{12}\,[E]^T + \psi_{22}\,[D]^T\} \qquad (40)$$

$$\psi_{11} = -\chi_{22}/\text{DET}$$

$$\psi_{12} = \chi_{12}/\text{DET}$$

$$\psi_{22} = -\chi_{11}/\text{DET}$$

$$DET = X_{11}\,X_{22} - (X_{12})^2$$

Using equs. (32) and (35)-(38) permits the expression of the incremental change in stress in terms of the strain increment and the negatives of the stresses existing at the time of the fracture, i.e.,

$$[\Delta\sigma] = \{ [C]_e - [E][H]^T - [D][M]^T \} [\Delta\varepsilon] - \{ \psi_{11}[E] + \psi_{12}[D] \} \Delta\sigma_{\theta_o} - \{ \psi_{12}[E] + \psi_{22}[D] \} \Delta\tau_{\theta_o} \tag{41}$$

Thus, for elastic behavior with one open crack, the incremental properties are:

$$[C]_{1c} = [C]_e - [E][H]^T - [D][M]^T \tag{42}$$

$$[L]_{1c} = - \{ \psi_{11}[E] + \psi_{12}[D] \} \Delta\sigma_{\theta_o} - \{ \psi_{12}[E] + \psi_{22}[D] \} \Delta\tau_{\theta_o} \tag{43}$$

If the value of Δu_n (equ. (37)) is such that the indicated total accumulated value of u_n is less than zero, then the interval must be appropriately reduced in length (i.e., appropriate value selected for "b", see Figure 8), so that at the end of the interval the crack just closes, see Table 1. If the value of separation at the beginning of the interval is $u_{n_{N-1}}$ and the value calculated from equ. (37) is Δu_n (where $u_{n_{N-1}} + \Delta u_n < o$), then the factor (1-b) of Figure 8 must be such that:

$$u_{n_{N-1}} + (1-b)\,\Delta u_n = 0$$

or $$b = \frac{u_{n_{N-1}} + \Delta u_n}{\Delta u_n}$$

Response with two open cracks: Figure 9 is a pictorial representation of a particle with two open fractures. It is apparent that as long as both fractures remain open that the particle can experience arbitrary changes in strain without any accompanying stress, hence the incremental stiffness is zero, i.e.,

$$[C]_{2c} = [0] \tag{44}$$

The remaining task is to calculate $[L]_{2c}$ (non-zero values would be due to the formation of one or the other of the cracks) and, for a given strain increment, the amount of deformation experienced by each of the fractures.

Because of the limitation that has been placed upon the permissible response modes, see Table 1, only one of the two fractures can be newly formed. If one of the two fractures is newly formed, denote the stress that existed in the particle prior to this occurence as $[\sigma]_o$. Upon the formation of the second open fracture the particle is no longer capable of carrying any stress, and thus $[L]_{2c}$ must be equal to the negative of $[\sigma]_o$, i.e.,

$$[L]_{2c} = - [\sigma]_o \tag{45}$$

This "release" of stress due to the formation of the crack produces an elastic change in strain (use inverse of equ. (10)), i.e.,

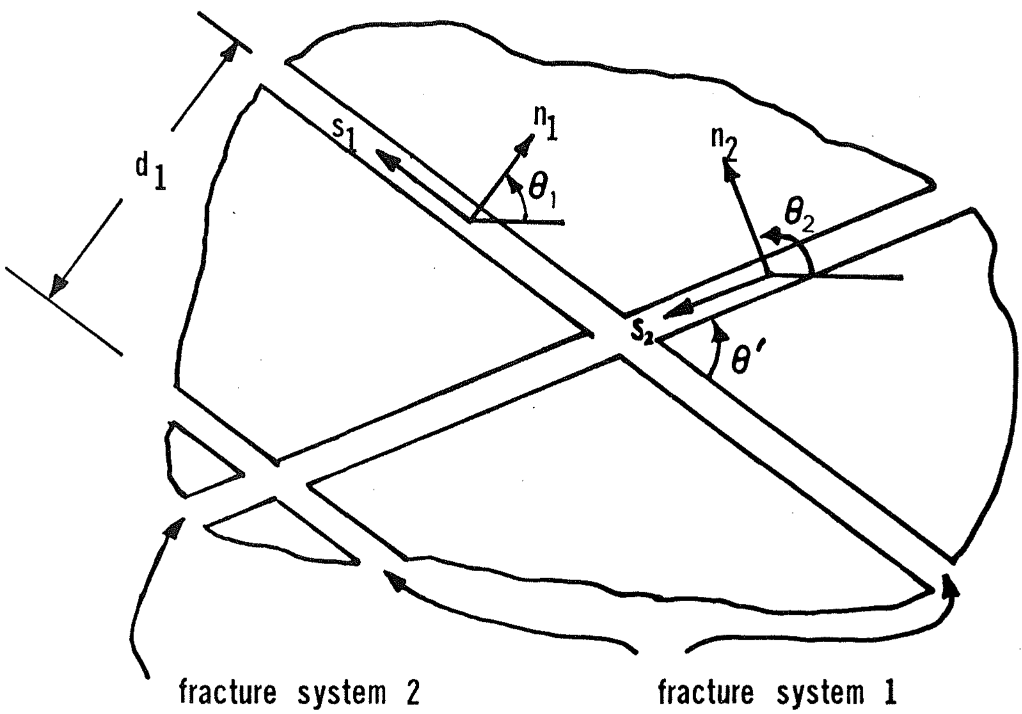

Figure 9 Pictorial Representation of Particle with Two Open Fracture Systems.

$$[\Delta\varepsilon]_o = - [C]_e^{-1} [\sigma]_o \tag{46}$$

At this point it is convenient to express all strain quantities in a coordinate system aligned with one of the fracture systems. Using equ. (12) the strain of equ. (46) is transformed to the n_1-s_1 system, i.e.,

$$[\Delta\varepsilon]_{o_1} = [T]_1^T [\Delta\varepsilon]_o$$

The imposed strain increment $[\Delta\varepsilon]$ is also transformed to the n_1-s_1 system, i.e.,

$$[\Delta\varepsilon]_1 = [T]_1^T [\Delta\varepsilon]$$

The total strain increment, $[\Delta\varepsilon]_1$, is equal to the sum of $[\Delta\varepsilon]_{o_1}$ and the strain produced by the deformations of the two fracture systems, i.e., Δu_{n_1}, Δu_{s_1}, Δu_{n_2}, and Δu_{s_2}. This latter strain is expressed by using equ. (27) and noting that, relative to the n_1-s_1 system, the angle used in expressing the deformation of the first fracture plane is zero and for the second is $\theta' = \theta_2 - \theta_1$, i.e.,*

$$\Delta\varepsilon_{n_1} = \Delta\varepsilon_{n_{1_o}} + \Delta u_{n_1} + \frac{1}{2}(1 + \cos 2\theta')\,\Delta u_{n_2} - \frac{1}{2}\sin 2\theta'\,\Delta u_{s_2} \tag{47}$$

$$\Delta\varepsilon_{s_1} = \Delta\varepsilon_{s_{1_o}} + \frac{1}{2}(1 - \cos 2\theta')\,\Delta u_{n_2} + \frac{1}{2}\sin 2\theta'\,\Delta u_{s_2} \tag{48}$$

$$\Delta\gamma_{n_1 s_1} = \Delta\gamma_{n_1 s_{1_o}} + \Delta u_{s_1} + \Delta u_{n_2}\sin 2\theta' + \Delta u_{s_2}\cos 2\theta' \tag{49}$$

---

* It is assumed that there is no interference produced by simultaneous deformation of two fracture systems.

Now values for $\Delta u_{n_1} \ldots \Delta u_{s_2}$ must be selected to satisfy equs. (47)-(49). Because this involves selecting four unknowns to satisfy three equations a non-uniqueness is apparent. Thus, an arbitrary assignment will need to be made for the value of one of the unknowns; because only rigid body motion is involved in this assumption, it does not affect the overall results.

The process of satisfying equs. (47)-(49) must yield values of Δu_{n_1} and Δu_{n_2} such that the total accumulated values of u_{n_1} and u_{n_2} remain greater than zero (if this is not possible then the end of the response interval is signified)*. Equs. (47)-(49) indicate two distinct possible modes of behavior, i.e., the cases of $\theta' = \pi/2$ and $\theta' \neq \pi/2$. This distinction is not, however, as clear as it might appear because when a finite number of significant figures are employed strict equality has no real meaning. What does have significance is the immediate neighborhood of the equality, i.e., $\theta' \approx \pi/2$; in accordance with the convergence limit used in the establishment of values of θ (Appendix A) this condition is defined as $\cos 2\theta' < -.9976$ (i.e., approximately $88° < \theta' < 92°$).

For the case of $\theta' \approx \pi/2$ (i.e., cracks approximately perpendicular), because $\sin 2\theta' \approx 0.0$, the last terms of equs. (47) and (48) are, for reasonable values of Δu_{s_2}, small and generally unimportant. Thus, for all practical purposes there is no arbitrariness in the selection of Δu_{n_1} and Δu_{n_2} as they are defined by equs. (47) and (48). Hence, the non-uniqueness is removed by

---

* Amoung the history items that must be calculated and stored for each particle are the values of u_n for each fracture system.

completely eliminating the last two terms of equs. (47) and (48) by requiring that $\Delta u_{s_2} = 0$; solving equs. (47)-(49) in this manner yields:

$$\Delta u_{n_2} = \frac{2}{1-\cos 2\theta'} [\Delta\varepsilon_{s_1} - \Delta\varepsilon_{s_{1_o}}] \quad (50)$$

$$\Delta u_{n_1} = \Delta\varepsilon_{n_1} - \Delta\varepsilon_{n_{1_o}} - \frac{1}{2}(1 + \cos 2\theta')\, \Delta u_{n_2} \quad (51)$$

$$\Delta u_{s_1} = \Delta\gamma_{n_1 s_1} - \Delta\gamma_{n_1 s_{1_o}} - \Delta u_{n_2} \sin 2\theta' \quad (52)$$

also $$\Delta u_{s_2} = 0 \quad (53)$$

For the case of $\theta' \neq \pi/2$ (as defined previously), a first attempt is made to remove the non-uniqueness by using equs. (50)-(53); if the predicted values of Δu_{n_1} and Δu_{n_2} do not close either of the cracks, the search is over. If, however, one or both of the cracks is predicted to close an attempt is made to remove the non-uniqueness in some way that does not produce this effect.

For the alternative selection, equs. (47) and (48) are added, i.e.,

$$\Delta u_{n_1} + \Delta u_{n_2} = \Delta\varepsilon_{n_1} - \Delta\varepsilon_{n_{1_o}} + \Delta\varepsilon_{s_1} - \Delta\varepsilon_{s_{1_o}} \quad (54)$$

If the result of the attempt to use equs. (50)-(53) predicts that crack "2" is the first to close, then the non-uniqueness is removed by arbitrarily requiring crack "2" to "very nearly" close, i.e., set

$$\Delta u_{n_2} = - u_{n_2} + \delta$$

where u_{n_2} = previous width of crack "2"

δ = very small number (e.g., 10^{-8})

Equation (54) now yields:

$$\Delta u_{n_1} = \Delta\varepsilon_{n_1} - \Delta\varepsilon_{n_{1_o}} + \Delta\varepsilon_{s_1} - \Delta\varepsilon_{s_{1_o}} + u_{n_2} - \delta$$

If this value of Δu_{n_1} is not sufficient to close crack "1" then the search is over. If, however, a closing of the crack is indicated it is arbitrarily required to very nearly close, i.e., $\Delta u_{n_1} = - u_{n_1} + \delta$, and the length of the interval is adjusted (i.e., a transition to a situation of one open crack is produced, and the appropriate value of b for equ. (28) is calculated) so that the solution of equ. (54) closes crack 2, i.e., $\Delta u_{n_2} = - u_{n_2}$.

In contrast to the assumption of the above paragraph, if the initial use of equs. (50)-(53) should predict a closing of crack "1" instead, then the steps of the previous paragraph are carried out with the subscripts 1 and 2 reversed.

For the case of $\theta' \neq \pi/2$, once values for Δu_{n_1} and Δu_{n_2} are determined, Δu_{s_1} and Δu_{s_2} may be calculated from equs. (49) and (47) or (48). As the model is now formulated, these values are not required and therefore are not computed.

Response when sliding occurs along a closed fracture zone: To determine whether or not sliding deformation will occur along a closed fracture, an appropriate sliding criterion must be established.

Conceptually this criterion is to be distinguished from the failure initiation criterion discussed previously (Jaeger-60, Murrell-65, Herget-70). A Mohr criterion is also used for the sliding criterion (other criteria have been proposed, e.g., see Murrell-65, Hobbs-70). This criterion is, however, defined by different values of σ_t, c, and f, i.e.,

$$\eta \, \tau_\theta = c_s - f_s \, \sigma_\theta \tag{55}$$

Where f_s is the coefficient of sliding friction and c_s is the residual cohesion (for the present study σ_{t_s} is taken to be zero, however, for the representation of ductile type fracture zones, it may be desirable to modify this assumption).

The utilization of similar criteria, for the phenomena of failure initiation and subsequent sliding, is important in order to be able to account for the fact that the transition from fracture to sliding deformation is not always easy to distinguish. The precise occurence of this transition is somewhat nebulous for the following reasons:

a) The fracturing process may proceed rather slowly; e.g., consist of the gradual linking up of numerous small cracks, e.g., see Handin-63, Herget-70.

b) Continued sliding may reduce f due to the breaking of asperities, e.g., see Goodman-72.

c) If the value of σ_θ is large, then a clean fracture may not take place, instead a zone of ductile type deformation develops (for which, obviously, $c_s >> 0$), e.g., see Handin-63, Murrell-65.

The transition from the fracture (equ. (18)) to the sliding criterion (equ. (55)) is accomplished by writing both as a single equation with variable coefficients c and f, i.e.,

$$\eta \, \tau_\theta = c(\beta) - f(\beta) \, \sigma_\theta \tag{56}$$

Where c and f are prescribed functions of a measure of damage (evidence of such damage dependence may be found in the works of Handin-63 and Hobbs-70). Obviously for $\beta = 0$, $c = c_o$ and $f = f_o$, and for very large values of β, $c = c_s$ and $f = f_s$; the variation of c and f for intermediary values of β is a matter of conjecture and must be the subject of future study. An example of the type of functions used by the authors for $c(\beta)$ and $f(\beta)$ are given in Figure 10.

The nature of the measure of damage, β, must also be the subject of future study. A preliminary consideration of experimental evidence and intuition suggests a function of the history of the normal pressure and the relative sliding displacement of the opposing faces of the fracture, i.e., $\beta = \beta \, (\sigma_\theta, u_s d)$. For the sake of simplicity, the following definition has been used in the current study:

$$\beta = \sum \alpha |\Delta u_s| \tag{57}$$

where $\alpha = \begin{cases} 1 & u_n = 0 \\ 0 & u_n > 0 \end{cases}$

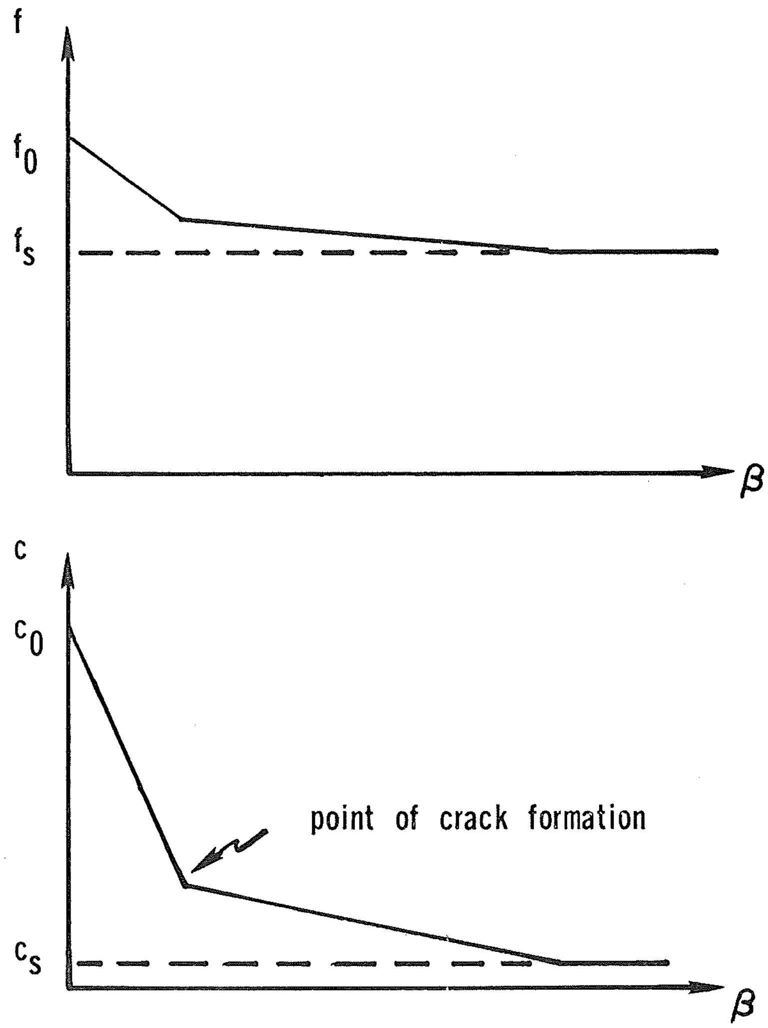

Figure 10 Typical Curves for c and f.

Thus, the measure of damage has been taken as the sum of the absolute values of the sliding deformation increments (Δu_s) that have occured when the crack is closed. This assumption is predicated on the concept that the greater the sliding deformation, the more complete is the fracture (break down of cohesion) and the smoother (due to polishing and grinding) the fracture faces. It would be more realistic to use the actual displacement $u_s d$ instead of the sliding deformation u_s; this would, however, require a prediction for the value of the fracture plane spacing d. The prediction of d for fractures along planes which are not natural planes of weakness needs further investigation.

In addition it appears that the value of σ_θ should play a more prominent role in the calculation of β (justification for this statement is given in a later section). In particular, indications are that large compressive values of σ_θ tend to prevent the formation of well developed fracture zones and, hence, for a given amount of sliding deformation should result in a reduction in the value of β (this trend would probably be reversed once a clean fracture has developed). Thus, one of the first revisions of the model that will be undertaken in the continuation of this research is to develop a more rational expression for the damage measure β.

Because it is possible to write both the fracture and sliding criteria in a single equation, there is no need, as long as the appropriate value of β is used, to distinguish between the formation of a new fracture and the sliding deformation of an existing closed fracture.

The initiation of sliding may occur at any point during increment N; once it has begun, due to the inherent instability caused by the decreasing values of c and f (see Figure 10), it is assumed that it continues to the end of the increment, see Table 1. The value of the damage at the beginning of the interval is denoted as β_b and at the end of the interval by β_N.

Equ. (56) must now be expressed in incremental form. It is first written in the form of equ. (20) (where c and f are appropriate functions of damage), i.e.,

$$[Q]^T[\sigma] + c = 0 \tag{58}$$

When the sliding begins equ. (58) yields:

$$[Q]_b^T [\sigma]_b + c_b = 0 \tag{59}$$

Where $[Q]_b$ (equ. (21)) makes use of the values of f_b (defined by β_b) and $[\sigma]_b$ (the stress state at the beginning of the sliding interval). At the end of the interval equ. (58) yields:

$$[Q]_N^T [\sigma]_N + c_N = 0$$

or

$$[Q]_b^T [\sigma]_N + [\Delta Q]^T [\sigma]_N + c_N = 0 \tag{60}$$

where

$$[\Delta Q] = [Q]_N - [Q]_b$$

Subtracting equs. (59) and (60) gives:

$$[Q]_b^T\,[\Delta\sigma] + [\Delta Q]^T\,[\sigma]_N + \Delta c = 0$$

or $$[Q]_b^T\,[\Delta\sigma] + \overline{\Delta}c = 0 \qquad (61)$$

where $$\overline{\Delta}c \equiv \Delta c + [\Delta Q]^T\,[\sigma]_N \qquad (62)$$

Equ. (61) is the incremental form of the failure-sliding criterion; its adequacy is based upon the assumption that the interval is sufficiently small so that within the interval the left-hand side of equ. (58) may be expressed as a linear combination of the values at the ends of the interval.

During the interval, the strain due to the elastic response of the rock is denoted by $[\Delta\varepsilon]'$ (expressed either by equ. (10) or (41)*; denote the appropriate matrix of incremental properties as $[C]'$). The remainder of the total strain $[\Delta\varepsilon]$ (denoted by a double prime) is then due to the sliding deformation of the fracture plane, i.e.,

$$[\Delta\varepsilon] = [\Delta\varepsilon]' + [\Delta\varepsilon]''$$

Using equ. (27) (note: any strain due to a possible open crack is taken care of by equ. (41), and hence is included in $[\Delta\varepsilon]'$):

$$[\Delta\varepsilon] = [\Delta\varepsilon]' + \Delta u_s\,[B]$$

---

* If equ. (41) is used because the open crack cannot be newly opened the [L] matrix is zero, see Table 1.

or

$$[\Delta\varepsilon]' = [\Delta\varepsilon] - \Delta u_s \, [B] \qquad (63)$$

The relationship between the stress $[\Delta\sigma]$ and the elastic strain $[\Delta\varepsilon]'$ is:

$$[\Delta\sigma] = [C]' \, [\Delta\varepsilon]'$$

Using equ. (62) to eliminate $[\Delta\varepsilon]'$ from the above expression yields:

$$[\Delta\sigma] = [C]' \, \{ [\Delta\varepsilon] - \Delta u_s \, [B] \}$$

Define:

$$[G] = - [C]' \, [B] \qquad (64)$$

Thus:

$$[\Delta\sigma] = [C]' \, [\Delta\varepsilon] + \Delta u_s \, [G] \qquad (65)$$

The magnitude of the sliding deformation, Δu_s, is determined by satisfying the incremental fracture-sliding criterion, i.e., equ. (61):

$$[Q]_b^T \, \{ [C]' \, [\Delta\varepsilon] + \Delta u_s \, [G] \} + \overline{\Delta}c = 0$$

or

$$\Delta u_s = - \xi \, \{ \overline{\Delta}c + [Q]_b^T \, [C]' \, [\Delta\varepsilon] \}$$

where

$$\xi = \frac{1}{[Q]_b^T \, [G]}$$

Because the right-hand side of this equation contains $\overline{\Delta c}$ which depends on $[\sigma]_N$, a quantity which is not known a priori, and because c_N and f_N are dependent upon the value of Δu_s (see equ. (62)), the value of Δu_s must be established by iteration.

Substituting the above expression into equ. (65) yields:

$$[\Delta\sigma] = [C]' \, [\Delta\varepsilon] - \xi \, [G] \, \{ \overline{\Delta c} + [Q]_b^T \, [C]' \, [\Delta\varepsilon] \}$$

or (using equ. (64))

$$[\Delta\sigma] = \{ [C]' - \xi \, [G] \, [Q]_b^T \, [C]' \} \, [\Delta\varepsilon] - \xi \, \overline{\Delta c} \, [G]$$

Thus the stiffness properties $[C]_s$ for the case of sliding of a closed fracture is:

$$[C]_s = [C]' - \xi \, [G] \, [Q]_b^T \, [C]' \tag{66}$$

The loss of strength $[L]_s$ due to reduction of sliding resistance along the fracture plane is

$$[L]_s = - \xi \, \overline{\Delta c} \, [G] \tag{67}$$

It needs to be emphasized that iteration is required to establish c_N, f_N and $[\sigma]_N$ which are needed in the calculation of $\overline{\Delta c}$. That is, the quantities Δu_s and $[\Delta\sigma]$ are estimated and used to predict c_N, etc.; these values are in turn used to produce Δu_s and $[\Delta\sigma]$ which are compared to the estimated values, etc.

Fragmentation of intact particles: Some fragmentation of sound rock may occur as a consequence of stress induced damage (see discussion in Section III-A-1). As this fragmentation takes place a certain amount of material is transferred from the intact phase to the rubble phase (see discussion in Section III-B-1). In the absence of sufficient experimental evidence to quantitatively describe this phenomenon, an arbitrary assumption is made for the purposes of this exploratory study. It is assumed, that during increment N, that the change in value of the relative proportion of particle i (i.e., P_i, see Figure 4) is expressed by the following equation (a negative value indicates loss of material due to fragmentation):

$$\Delta P_i = P_i \, M_x \, (e^{-D_r \gamma_n} - e^{-D_r \gamma_{n-1}}) \qquad (68)$$

Where: D_r - a prescribed parameter measuring ease of fragmentation

γ - a measure of the damage that produces fragmentation

M_x - the maximum fraction of material that may be lost from the particle due to fragmentation (currently in the model $M_x = 0.05$).

The fragmentation damage measure γ was selected to be equal to $(\frac{.05}{D_r})$ times the number of fractures in the particle plus the sum of the measures of damage β (see equ. (57)) for each of these fractures. This definition of γ was selected in an attempt to account for the fragmentation which occurs when the fracture first forms, plus any additional fragmentation which results from sliding deformation of the closed joint.

The arbitrary nature of the above expression is apparent, and thus, at this time, use of this feature should be restricted to exploratory studies.

Determination of particle behavior: At this point, the equations have been established which describe the various particle behavior mechanisms listed in Table 1. Thus the $[C]_{i_a}$, $[C]_{i_b}$, $[L]_{i_a}$, $[L]_{i_b}$ matrices which describe the response of the particles for each of the two intervals represented in Figure 8 and which enter into equs. (28) and (29) are given by equ. (10) or (42) or (44) or (66) and by (10) or (43) or (45) or (67). The factor b, which defines the boundary between the two intervals of Figure 8 and which enters into equ. (28), is determined from either equ. (26) or from the considerations relative to the closing of an open fracture.

The one item that remains unexplained is the description of the sequence of steps for determining which of the behavioral modes described in Table 1 is to be used for a given particle. The logic of this selection process is shown in Figure 11.

4. Composite Behavior:

The calculation of the incremental properties for the representative volume utilizes the results for the rubble phase, i.e., equ. (9), the results for the several intact phases, i.e., equs. (28) and (29), and equs. (6) and (7).

It is to be kept in mind (see discussion in Section II-B) that all the calculations of the previous sections are based upon an assumed knowledge of the strain increment $[\Delta\varepsilon]_N$ which, of course, is not usually known a priori. Hence, the composite

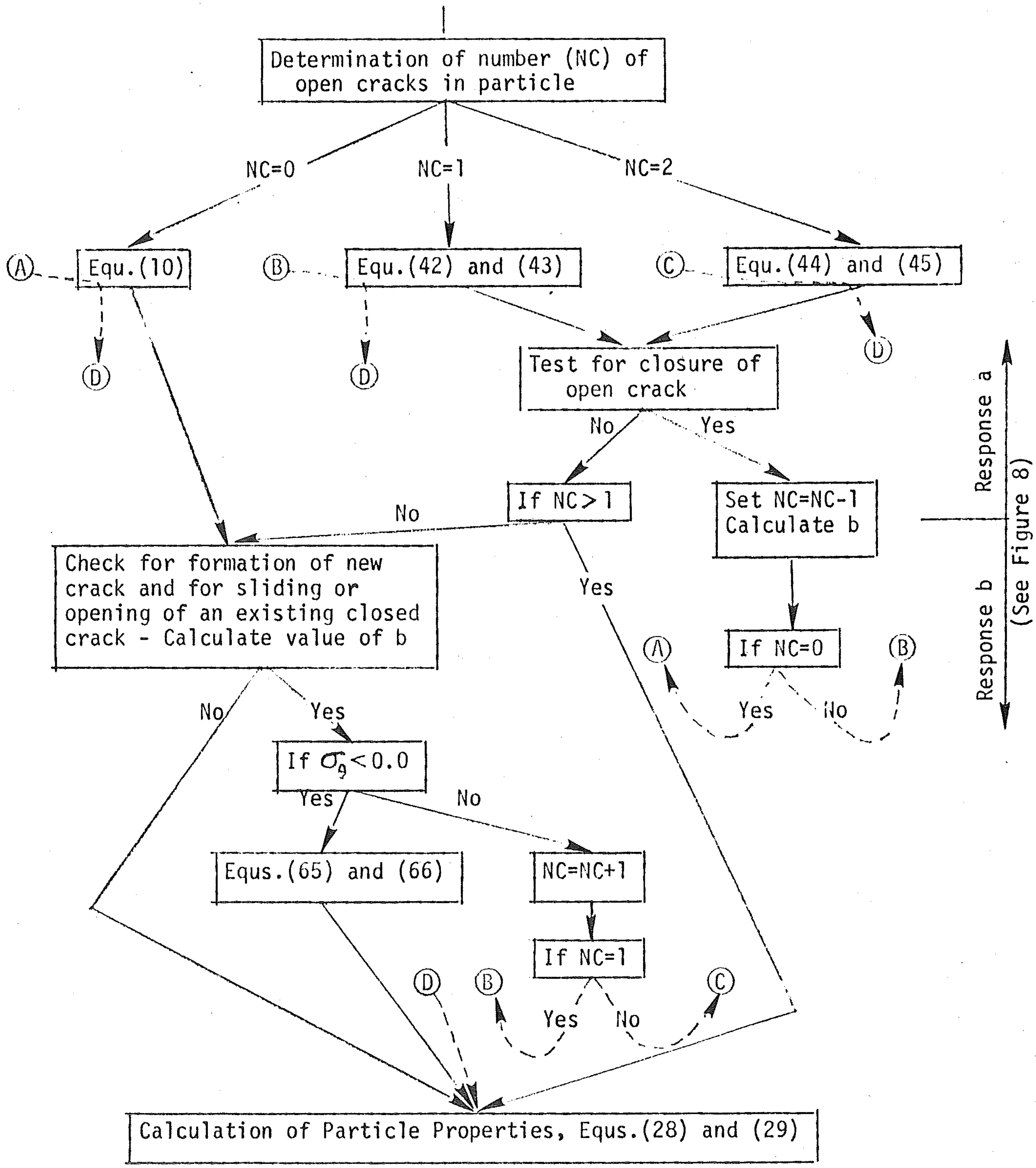

Figure 11 Logic for Selecting Particle Response Mode

(See Table 1)

properties for a given increment must be established in an iterative fashion.

C. COMPUTER PROGRAMS FOR MODEL:

It is apparent from the description of the model that numerical evaluation is not a simple operation and thus, even for the simplest cases, requires the use of a high speed computer.

Two types of applications of the model are of interest. Initially it is of particular importance to be able to use the model to predict results for simple laboratory tests. The second type of application is the primary use for which the model was developed (see discussion in Section II-B), i.e., to provide the incremental properties for finite element analyses of rock structures. Both of these needs were taken into account during the development of the two computer programs described in this section.

A computer subroutine (called PROP) has been prepared for the evaluation of the incremental properties of the representative volume for rock masses. This subroutine is so formulated that it may, with little difficulty, be directly incorporated into existing finite element programs for use in plane strain analyses of rock structures. The required characteristics of the finite element program are discussed in Section II-B. In addition, a small program for predicting the results of simple laboratory tests is given (called EVAL). This program makes use of the subroutine for the calculation of the properties of the rock. The use of these two programs is described in the remainder of this section.

1. Subroutine PROP:

The purpose of this subroutine is to calculate incremental properties for rock masses (based on the analysis of the representative volume). Throughout the development of this subroutine, the authors drew upon their extensive experience in the development and use of finite element programs, in an attempt to make the subroutine as compatible as possible with the more advanced of the existing two-dimensional finite element programs. While, to date, the authors have not had the time to incorporate the subroutine into one of their existing programs, it would be a relatively simple operation and is high on the agenda for the continuation of this research.

The logic of the subroutine follows very closely that presented in Figure 11, and thus does not require further discussion. In order to simplify this exploratory investigation, certain sophistications regarding dimension changes, dimension checks, print-out concerning the nature and scope of damage in the rock, etc., have not been included, but instead left for future work.

The listing of the program is given in Appendix B. The necessary steps for incorporating the subroutine into a finite element program are now described.

The call to the subroutine is:

CALL PROP (INTR, NELM, ISYMN, IC, DESP, CS, XLS)

The first four arguments are integer variables; the dimensions of the three arrays are DESP(3), CS(3,3) and XL(3).

Two preliminary calls must be made to the subroutine in order to enter material properties and to initialize history and property arrays for each of the finite elements. The formats for these calls are:

1) The first call is for the purpose of causing the subroutine to read the sets (one for each different type of rock) of parameters which describe the characteristics of the representative volume. The reading and echo printing of these parameters is done within the subroutine; the formats for the read statements are described later. For this call the only argument that has significance is INTR which must be assigned the value of 1.

2) The second call is for the purpose of establishing the auxiliary storage arrays for the properties and damage history of each finite element. This call must be executed for each of the rock elements, i.e., NELM times (the precise nature of the sequencing of these calls is described later). For this call the arguments of interest are (the material number must correspond to the appropriate rock type identification number described in the first call):

   INTR = 2

   NELM = Total number of rock elements

   IC = Material number of element

After these two preliminary calls are completed the subroutine is ready for the prediction of the incremental properties for each of the rock finite elements; these predictions take place within the incremental-iterative analysis loops (see discussion in Section II-B).

For each iteration of each increment the subroutine is called NELM times; the definitions of the arguments in the call are:

INTR = 3

NELM = Total number of rock elements

ISYMN = $\begin{Bmatrix} 0 \\ 1 \end{Bmatrix}$ if the subroutine is to return the material properties in $\begin{Bmatrix} \text{unsymmetrical form, equ. (1)} \\ \text{symmetrical form, equ. (2)} \end{Bmatrix}$

IC = Iteration number (note: the first iteration of each increment must be numbered 1, i.e., IC≥1)

DSEP = Estimate of strain increment, i.e., $[\Delta\varepsilon]_N$ of equ. (1)†

CS = Predicted incremental stiffness properties, i.e., $[C]_N$ of equ. (1) or $[C_s]_N$ of equ. (2)

XLS = Predicted incremental strain independent stresses i.e., $[L]_N$ of equ. (1) or $[L_s]_N$ of equ. (2).

The first five arguments are, of course, values provided to the subroutine by the calling program, whereas, the last two are incremental properties calculated within the subroutine and returned to the main program.

Within each iteration PROP must be called for each of the NELM elements; these calls must be in precisely the same order as was the case for the 2nd preliminary call.

The input of material parameters (read by the subroutine

† The current development is for plane strain, thus the three components of $[\Delta\varepsilon]_N$ are $\Delta\varepsilon_{x_N}$, $\Delta\varepsilon_{y_N}$, and $\Delta\gamma_{xy_N}$.

during the first preliminary call) is according to the following format:

A. Control Card (I5):

Col. 5: NOMAT (≤ 4)* -number of different types of rock

B. NOMAT sets of cards describing the rock types

1. Initial Card (4I5,E10.3)

Col. 5: MN (≤ 4) -material number (i.e., rock type identification number)

10: NONP (≤ 2)** -number of natural planes of weakness

14-15: NODIS (≤ 20) -number of intact particles (denoted by the symbol I in Section III-B-1)

20: $\text{ITYP} = \begin{cases} 0 & \text{isotropic rock} \\ 1 & \text{orthotropic rock} \end{cases}$

21-30: D_r-parameter used to control fragmentation of of intact particles (see equ. (68)).

---

* Indicates dimension limit.

** The limit of 2 for the number of natural planes of weakness is in contradiction to the final decision concerning this limit (i.e., 3) as stated in the main body of the report; time did not permit the modification of the subroutine to reflect this decision; in the authors' judgment there should be very few instances when this will be a detriment. In addition, contrary to the statement in the main body of the report, time did not permit the inclusion of residual stress effects.

2. Elastic Properties Card (8E10.2) - see equ. (1)

ITYP Cols	0	1
1-10	E	C_{11}
11-20	ν	C_{12}
21-30		C_{13}
31-40		C_{22}
41-50		C_{23}
51-60		C_{33}
61-70		ϕ (in degrees)*
71-80		R (see equ. (16))

3. NONP+1 Groups of cards giving strength parameters (first NONP are for the natural planes of weakness, the last group is for the sound rock):

a) 1st card (2I5,2E10.3):

Col. 5: NOF (≤ 5) -number of points defining the function $f(\beta)$ (see Figure 10)

10: NOC (≤ 5) -number of points defining the function $c(\beta)$ (see Figure 10)

11-20: σ_t-tensile strength

21-30: θ -orientation (in degrees) of natural plane of weakness (see Figure 6); left blank for sound rock

* The C_{11}, C_{12}, ... C_{33} properties are given for the preferred directions x_1-x_2, the angle ϕ is measured from the x axis to the x_1 axis.

b) As many cards (8E10.2) as needed to describe the function $c(\beta)$ (see Figure 10)

Col. 1-10: c_1

11-20: β_1

21-30: c_2

.
.
.

β_{NOC}

c) As many cards (8E10.2) as needed to describe the function $f(\beta)$ (see Figure 10)

Col. 1-10: f_1

11-20: β_1

21-30: f_2

.
.
.

β_{NOF}

4. Description of Integrity Factor - As many cards (8E10.2) as needed to give the distribution function for the integrity factor F (see Figure 4)*:

Col. 1-10: P_1

11-20: F_1

21-30: P_2

.
.
.

F_{NODIS}

This completes the input of the parameters which describe

---

* The proportion of the rubble phase is automatically assigned, i.e., $P_{NODIS+1} = P_{I+1} = 1 - \sum_{i=1}^{NODIS} P_i$.

the representative volume.

The subroutine makes use of one auxiliary storage unit in a sequential operation. In the subroutine this unit is labeled "1" and treated as a tape unit (it can, of course, be a disk simulation of a tape unit). Special provision is made so that this auxiliary storage is not used when NELM = 1, i.e., when the subroutine is used in the analysis of simple laboratory samples. The storage required by the subroutine for COMMON/B1/ may be used by the main program, for other purposes, prior to the 2nd preliminary call; the storage for the second common block is required by the subroutine for the period beginning with the first preliminary call and ending with the completion of the second preliminary call.

2. Program EVAL:

In order to evaluate the effectiveness of the proposed characterization for the structural properties of rock, that is embodied in subroutine PROP, it is necessary to incorporate the subroutine into an analysis program. Because the preliminary evaluation was limited to simple laboratory tests it was not necessary to use a general finite element program. For the sake of economy the following special analysis program applicable to simple laboratory tests was developed.

The program is capable of considering simple three-dimensional states; its use in this phase of the research is, however, limited to the plane strain case. The program can treat any combination of prescribed histories of the six stress and/or strain quantities, i.e., prescriptions of the histories of (σ_x or ε_x) and (σ_y or ε_y)

and . . . (τ_{yz} or γ_{yz}). The only restriction placed upon these histories is that during the course of the analysis the prescribed quantities cannot be changed, e.g., at some point in the analysis a change from a prescription of σ_x to a prescription of ε_x is not permissible. During the last stages of this research effort this restriction was found to be somewhat inconvenient and will be removed early in the next phase of the research program.

Denote two of the six quantities to be described (e.g., σ_x or ε_x, and σ_y or ε_y) as f and g. Now, although time is not explicitly involved in the characterization, and hence in the analysis, for the purpose of describing the stress and strain histories it is convenient to introduce it. Thus the prescription of the histories of f and g can be thought of as a prescription of the functions f(t) and g(t). For the purpose of this analysis the histories are described by a series of linear segments, e.g., see Figure 12. The histories are prescribed by successively giving values for f_1, g_1; f_2, g_2; etc.

The effects of friction between the testing machine and the sample may be included in the analysis in a very crude fashion by requiring that a stress be developed to oppose strain. This option is invoked by prescribing non-zero values for the environmental stiffness factors, ENVSTF(I). For example, if τ_{xy} is prescribed to be h(t) and ENVSTF(4) = k_4 then the actual shear stress which is developed is $h(t) - k_4\gamma_{xy}$. The last term may be used to crudely simulate the development of a shear stress, due to friction, on the ends of the sample.

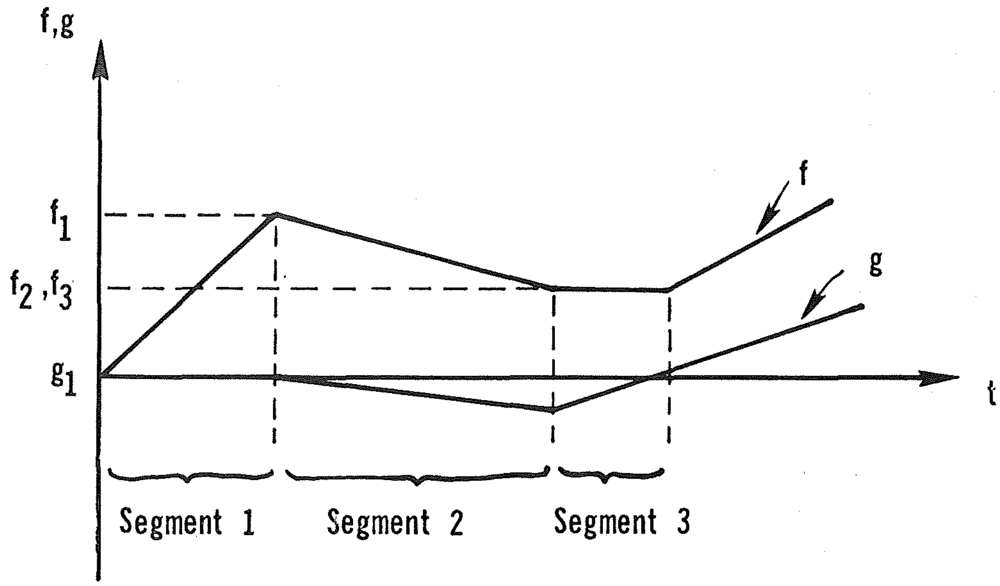

Figure 12 Examples of Prescribed Functions.

Program EVAL is based on a standard incremental-iterative analysis procedure for nonlinear problems. This procedure is well documented in many places (e.g., see Taylor-72) and thus need not be described here. Within each iteration, program EVAL calls subroutine PROP for a prediction of the incremental structural properties of the rock. The required degree of convergence is controlled by the specified maximum value (CONLNT) for the average error in the strain increment estimate used in the calculation of the incremental properties. The rate of convergence may be improved by using an appropriate value for the convergence factor "CONFAC"; the improved estimate of the strain increment is written in the form:

$$\widehat{[\Delta\varepsilon]}_N = [\Delta\varepsilon]_{N-1} + \text{CONFAC} \left\{ [\Delta\varepsilon]_N - [\Delta\varepsilon]_{N-1} \right\}$$

If the iteration scheme fails to converge within twenty iterations the program automatically halves the increment, and the iteration scheme is repeated. In a given analysis, if this halving procedure takes place more than fifteen times, it is taken as an indication that the problem is unstable and the analysis is aborted with the statement, "Sample is no longer stable". Once the halving procedure has been used, if the iteration scheme at any time converges in less than four iterations, the increment is doubled for the next step.

The listing of the program is given in Appendix B; the use of the program is described below:

A. Title Card (12A6):

Col. 1-72: Information to be printed as the heading of the output

B. Control Card (6I1, I4, 6E10.2, 2F5.1):

Col. 1: IFF(1) = $\begin{Bmatrix} 0 \\ 1 \end{Bmatrix}$ if $\begin{Bmatrix} \sigma_x \\ \varepsilon_x \end{Bmatrix}$ prescribed

2: IFF(2) = $\begin{Bmatrix} 0 \\ 1 \end{Bmatrix}$ if $\begin{Bmatrix} \sigma_y \\ \varepsilon_y \end{Bmatrix}$ prescribed

3: IFF(3) = $\begin{Bmatrix} 0 \\ 1 \end{Bmatrix}$ if $\begin{Bmatrix} \sigma_z \\ \varepsilon_z \end{Bmatrix}^{*}$ prescribed

4: IFF(4) = $\begin{Bmatrix} 0 \\ 1 \end{Bmatrix}$ if $\begin{Bmatrix} \tau_{xy} \\ \gamma_{xy} \end{Bmatrix}$ prescribed

5: IFF(5) = $\begin{Bmatrix} 0 \\ 1 \end{Bmatrix}$ if $\begin{Bmatrix} \tau_{xz} \\ \gamma_{xz} \end{Bmatrix}$ prescribed

6: IFF(6) = $\begin{Bmatrix} 0 \\ 1 \end{Bmatrix}$ if $\begin{Bmatrix} \tau_{yz} \\ \gamma_{yz} \end{Bmatrix}$ prescribed

10: ITYPE = $\begin{Bmatrix} 0 \\ 1 \end{Bmatrix}^{**}$ for $\begin{Bmatrix} \text{2-D} \\ \text{3-D} \end{Bmatrix}$ PROP Subroutine

11-20: ENVSTF(1)
21-30: ENVSTF(2)
.
.
.
61-70: ENVSTF(6)
} environmental stiffnesses

71-75: CONFAC - convergence factor

76-80: CONLMT - convergence criterion

C. Material Properties - At this point the properties of the rock are supplied to Subroutine PROP (see description of input for PROP)

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* For plane strain IFF(3) = IFF(5) = IFF(6) = 1 and the histories of ε_z, γ_{xz}, γ_{yz} are prescribed to be zero.

** The use of the current subroutine PROP requires that ITYPE = 0.

D. History Cards (1X, I4, 6E10.2), for each history segment (see Figure 11) the following card must be supplied

Col. 2- 5: NMIS - minimum number of increments to be used in this segment*

6-15: Value of σ_x or ε_x** at end of segment

16-25: Value of σ_y or ε_y at end of segment

26-35: Value of σ_z or ε_z at end of segment

36-45: Value of τ_{xy} or γ_{xy} at end of segment

46-55: Value of τ_{xz} or γ_{xz} at end of segment

56-65: Value of τ_{yz} or γ_{yz} at end of segment

E. End Card (I1) - Number 9 punched in Col. 1.

The above sequence (A-E) of cards is repeated for each analysis and placed in the data deck consecutively.

D. NUMERICAL RESULTS:

Some representative results predicted by the model, for simple stress and strain histories, are given in this section. These results were obtained by using the two programs described in the previous section.

Unfortunately, a state of plane strain is very difficult to achieve in the laboratory, and thus no experimental results for this state were located in the literature. In order to develop some feel

---

* The automatic halving process previously described may result in the use of a greater number of increments.

** The determination of which of the values of σ_x or ε_x is being supplied depends upon the value of IFF(1), etc.

for the appropriateness of the proposed model, the results are qualitatively compared to experimental results from cylindrical tri-axial tests. There is, of course, no reason to expect quantitative agreement between these two states, but one would expect similarities in their general characteristics. Thus, if the model can be made to predict results which are similar in nature to measured cylindrical tri-axial data, it is a strong indication that the model has the capabilities of predicting real rock behavior. From the large number of predictions made with the model, a few of the most interesting are briefly discussed below.

Of the experimental results that the authors located in the literature, those due to Wawersik-70 appear to be the least influenced by the characteristics of the experimental apparatus (in light of recent studies of the importance of end effects in tests of concrete, Kupfer-69, the authors suggest that even these results may be rather highly influenced by such disturbances). Figure 13 reproduces from Wawersik's paper a series of stress-strain curves taken with different confining pressures. Values for the several parameters which define the model (see Section III-C-1) were selected so as to give a stress-strain curve (zero confining pressure), for plane strain, that qualitatively agrees with the one given in Figure 13 for cylindrical tri-axial conditions. Because of the quantitative meaninglessness of comparing plane strain and cylindrical tri-axial results, no attempt was made to select the parameters to obtain good quantitative agreement. The values of the model parameters are given in Figure 14; the predicted behavior for zero confinement pressure is given in Figure 15. It

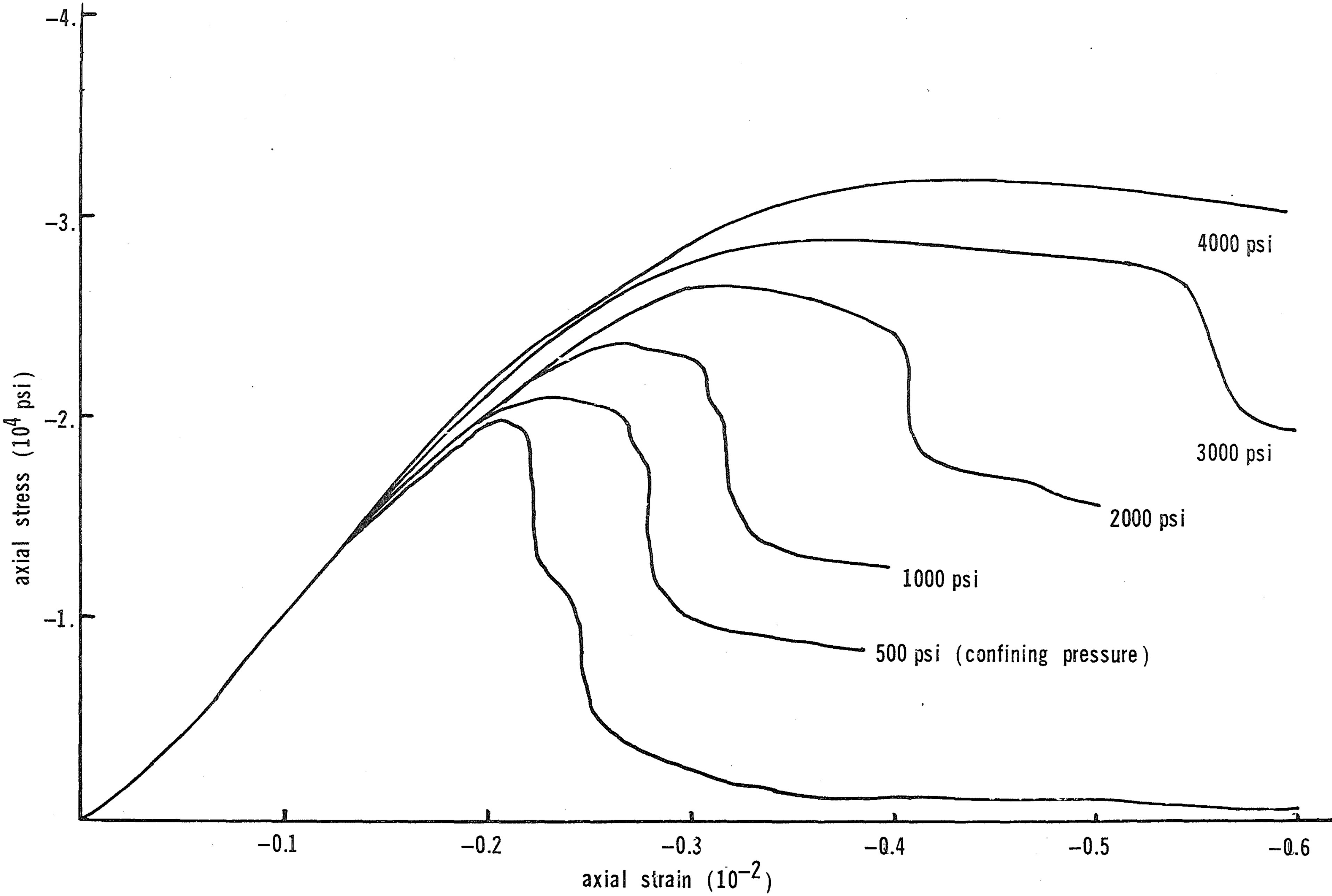


Figure 13 Experimental Results Taken from Wawersik - 70 for Cylindrical - triaxial Tests of Tennessee Marble II.

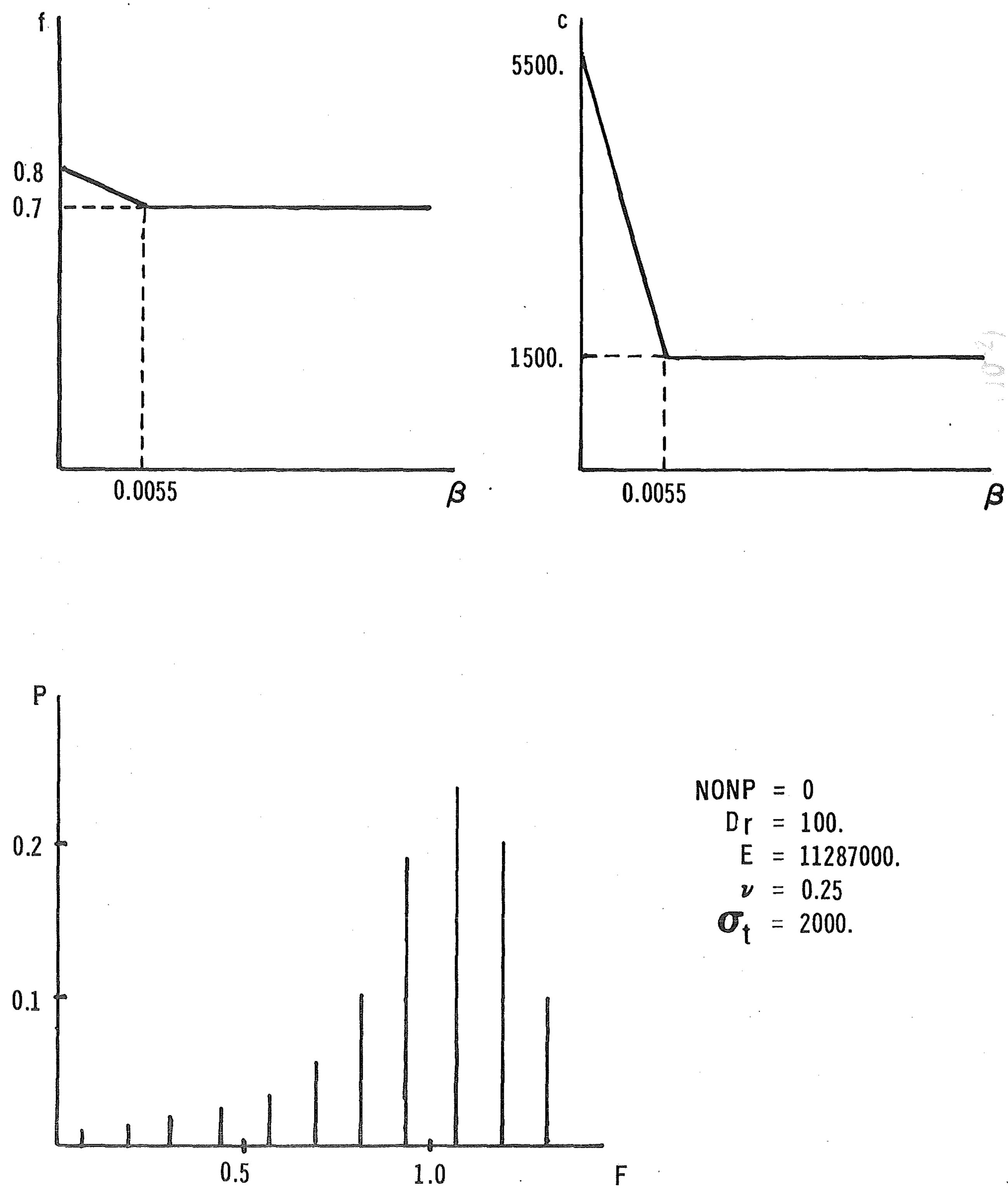


Figure 14 Model Parameters Used in Study of Tennessee Marble II.

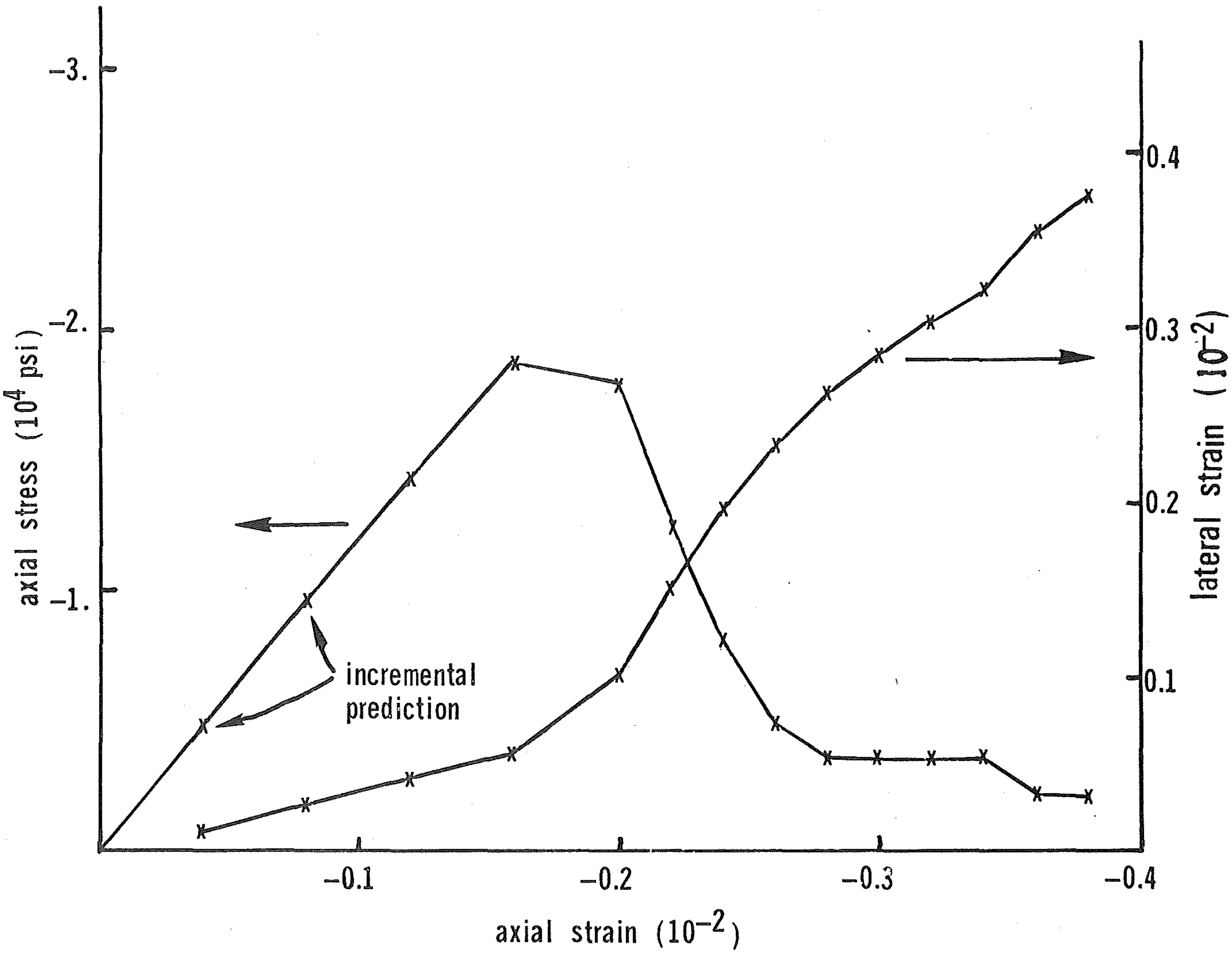


Figure 15 Predicted Stress – strain and Lateral Strain – axial strain Curves (Zero Confining Pressure – State of Plane Strain) for Tennessee Marble II.

must be emphasized that the parametric values, given in Figure 14, were selected to simulate cylindrical tri-axial results with a plane strain model, and thus are not correct values for the rock (i.e., values that would be used for actual plane strain conditions). When viewing these results, two facts should be kept in mind. First, the detailed shape of the descending portion of the experimental curve may be as much a product of the test arrangement, sample properties and test equipment as of the fundamental characteristics of the rock. Second, the local variations in this portion of the analytical curve are, to some extent, dependent upon the sizes of the increments used in the analysis. Thus, once the peak strength is exceeded, only the general nature of the curves are of interest.

The comparison of these results suggest that the model has the capability of capturing the general one-dimensional stress-strain characteristics of this particular rock. While Wawersik does not report any lateral strain (or volume change) measurements, the predicted results are in good qualitative agreement with observations made by other experimentalists, e.g., see Walsh-65 and Bieniawski-67_b.

Utilizing the same parameters, Figure 14, the predictions given in Figure 16 for pressurized plane strain tests were obtained. A comparison of Figures 13 and 16 illustrates a shortcoming of the model which was previously alluded to in Section III-B-3. The model's prediction of strength loss (descending portion of stress-strain curve) is a consequence of the reduction in c and f values (see Figure 14) as a function of β. The poor correlation between Figures 13 and 16, for large values of confining pressure, is explainable by

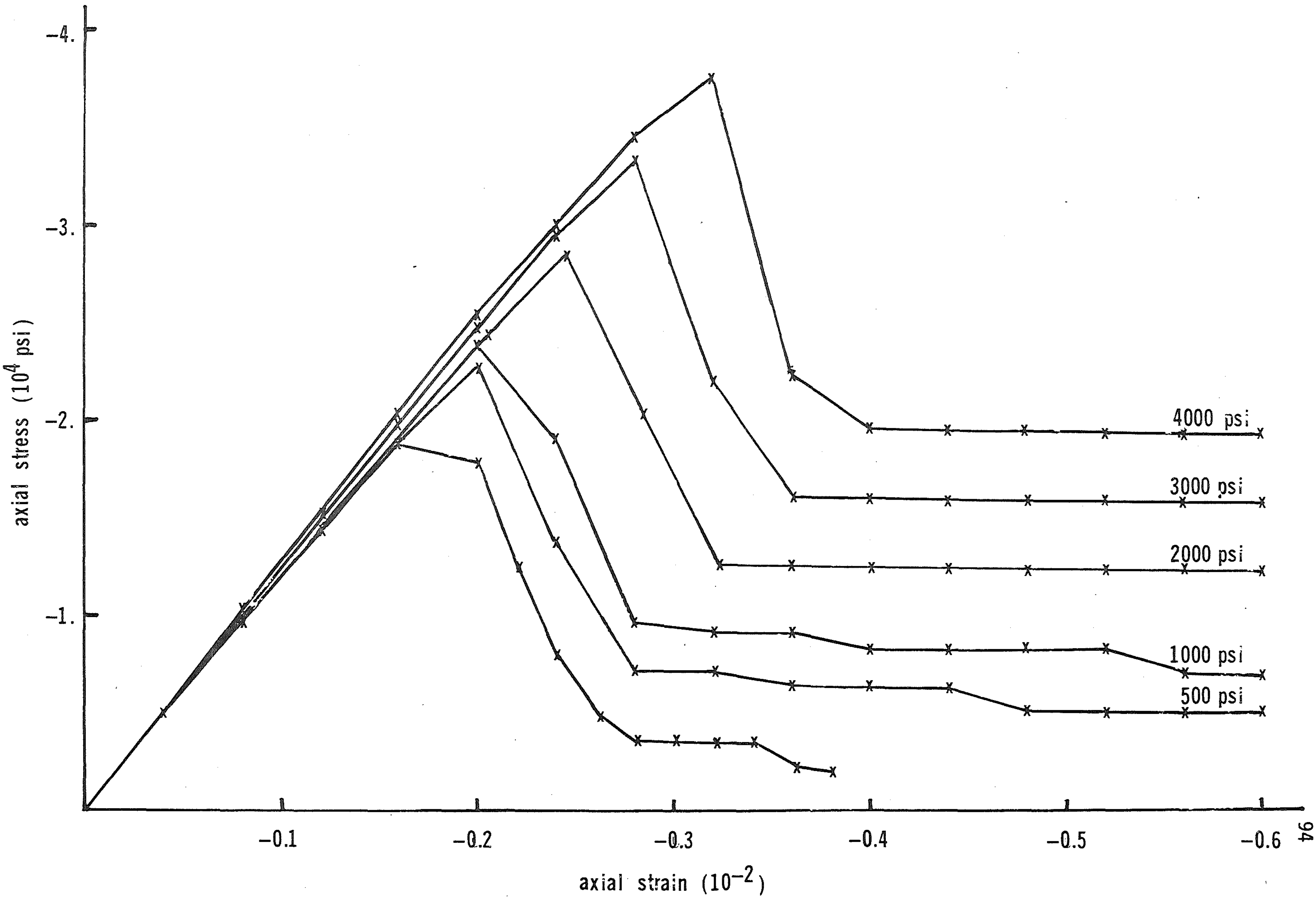


Figure 16 Model Predictions for Confined Plane – strain Response of Tennessee Marble II.

the fact that the model, as now formulated, does not account for an apparent decrease in the damage rate for high confining pressures. An indication of the validity of this assertion is found in the following study.

The c and f curves (Figure 14) used in the prediction of the results reported in Figure 16 were modified so as to make them decrease less rapidly with increases in the value of β, see Figure 17 (in actuality what is needed is to modify the definition of β, equ. (57), so that for higher mean pressures it accumulates more slowly). The revised predictions, for 2000 psi confining pressure, are shown in Figure 18. With further revision, the predicted results could have been brought even more closely into line with the observations. A comparison of Figures 13, 16 and 18 demonstrate the importance of the proposal to make the measure of damage (β) a function of the mean pressure.

Wawersik also gave experimental results for the case when the load is cycled from a point on the descending portion of the curve, see Figure 19. Predicted results for a single such unloading-reloading cycle are shown in Figure 20. The remarkable similarity of these results is particularly significant when viewed in light of the complexities of the nonlinear inelastic behavior of the region beyond peak strength. The fact that the unloading-reloading curve of Figure 20 is "open" instead of "closed" as in Figure 19 is apparently, at least in part, due to the fact that finite sized incremental steps are used in the prediction process.

It is also of interest to consider the predicted behavior

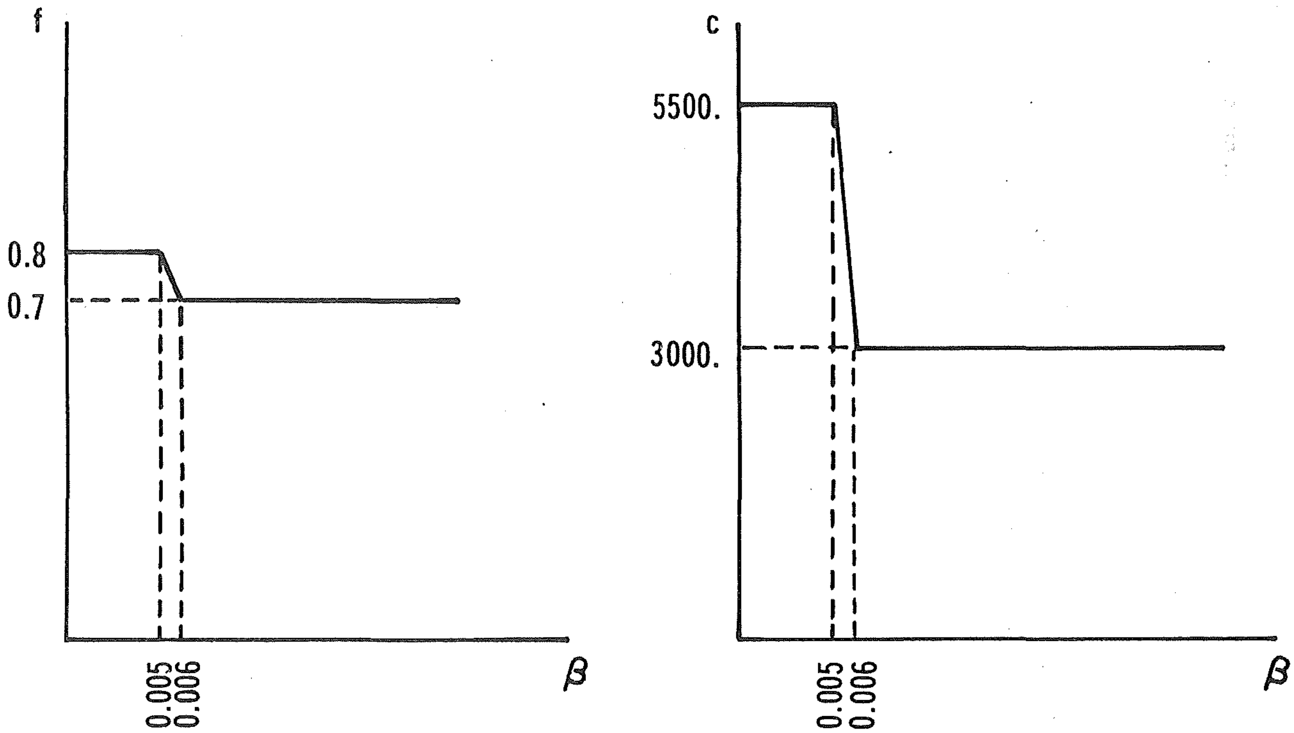


Figure 17 f and c Functions Modified to Account for High Confining Pressure – Tennessee Marble II.

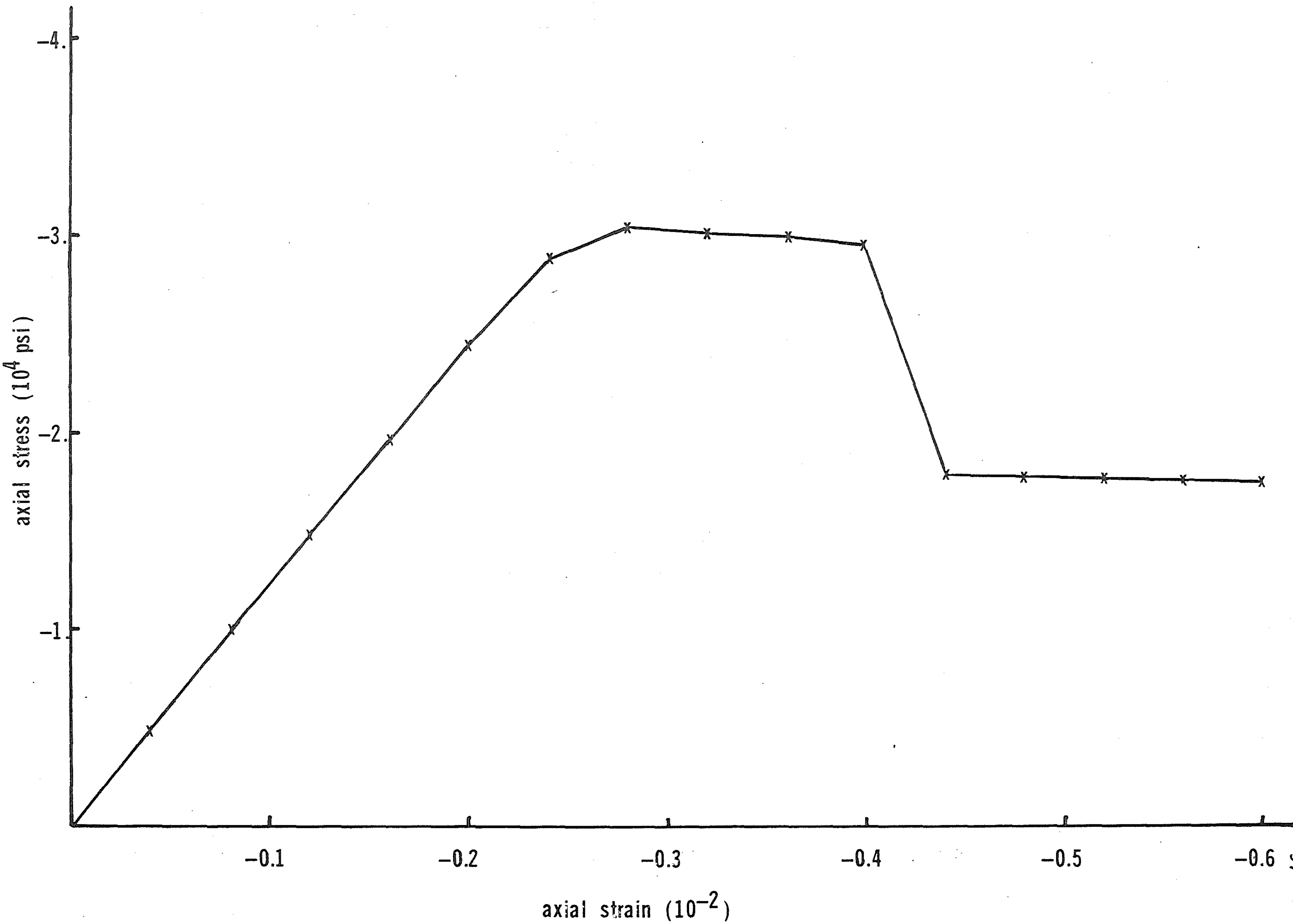


Figure 18 Modified Model Predictions for Confined (2000 psi) Plane – strain Response of Tennessee Marble II.

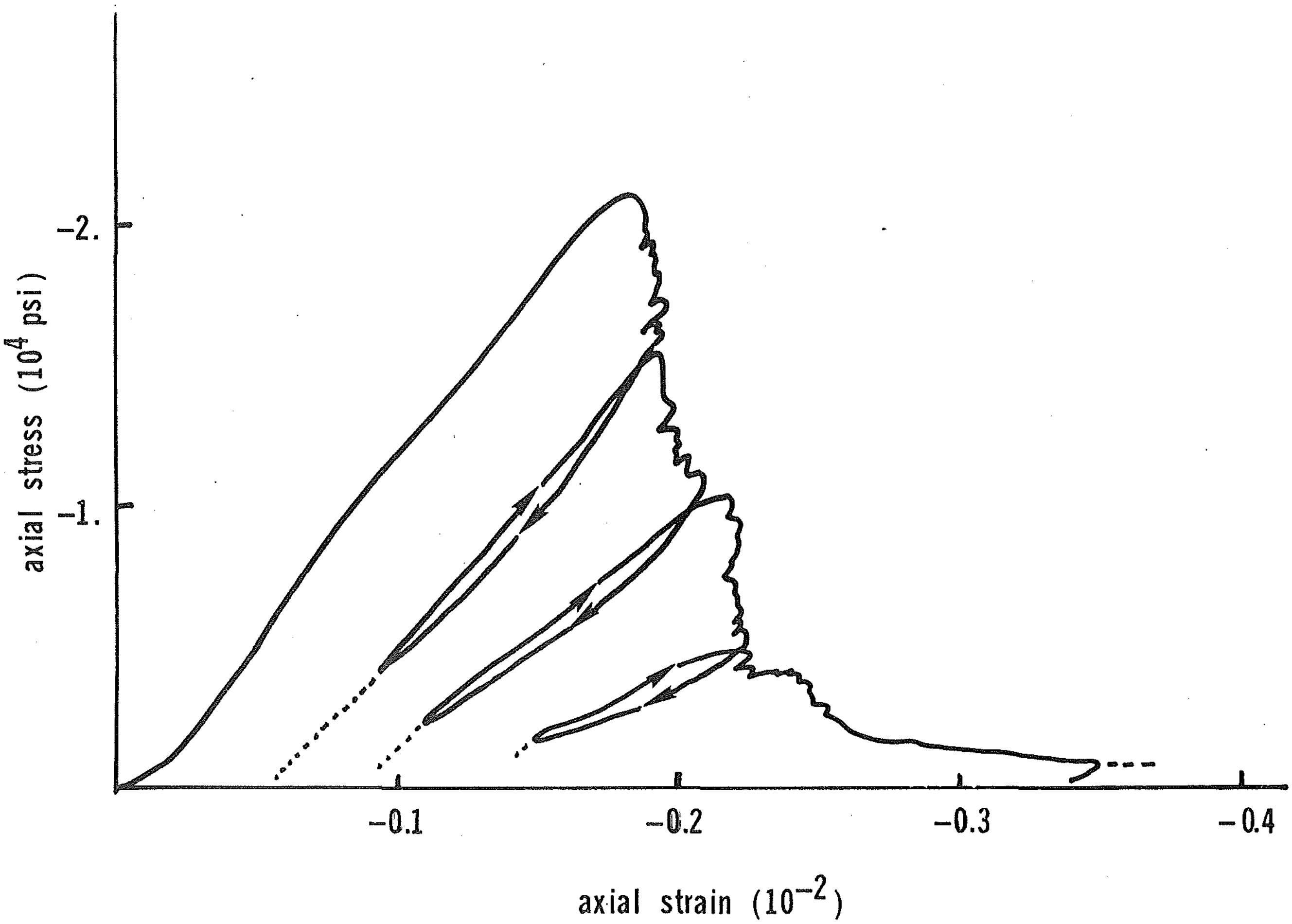


Figure 19 Experimental Results Taken from Wawersik – 70 for Unloading – reloading Test of Tennessee Marble II.

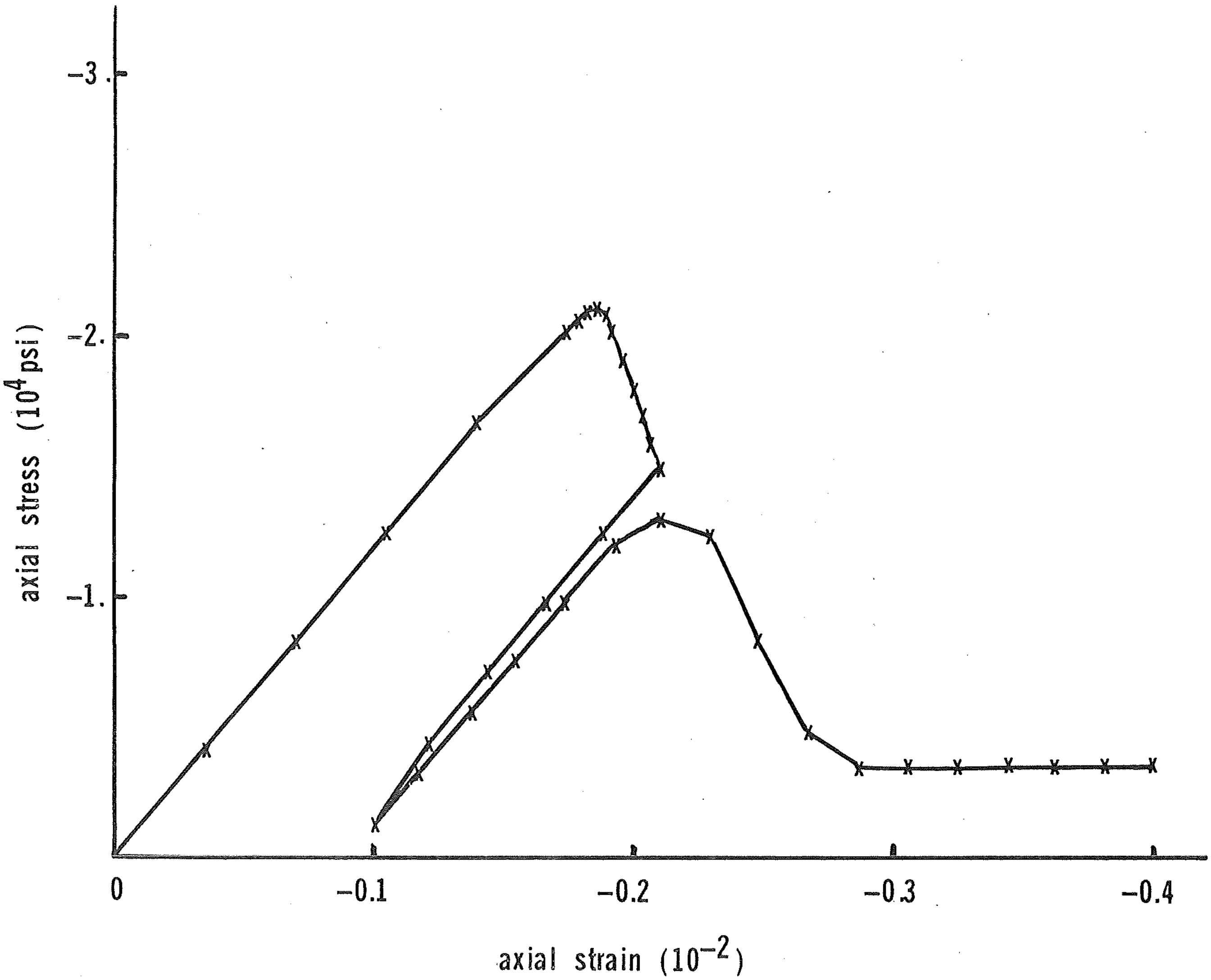


Figure 20 Model predictions for unloading and reloading of Tennessee Marble II

for the case where the unloading-reloading takes place prior to peak strength, see Figure 21. In accordance with experimental observations, (see Section III-A-1) a small but observable hysteresis loop is present (a result of damage to some of the particles).

In his experimental study, Wawersik observed two fundamentally different types of behavior, i.e., somewhat ductile and catastrophically brittle (he referred to rocks exhibiting these behaviors as Class I and Class II, respectively). Tennessee Marble II (Figure 13) is of the first type and Solenhofen Limestone of the second. The uniaxial stress-strain curve obtained by Wawersik for the limestone is reproduced in Figure 22; utilizing the parameters given in Figure 23, the results shown in Figure 24 were predicted with the model.

Because of the limited number of different stress and strain histories considered, and because in fact the experimental results were not for plane strain, any conclusions drawn from the above comparisons of model predictions with experimental observations must be treated as very tentative.

Nonetheless from the results presented herein, and others not reported, it would appear justifiable to state that the model has the demonstrated capability of capturing, with a fair degree of accuracy, the behavior of rock for at least relatively simple stress and strain histories; and that these capabilities will be significantly enhanced once certain modifications are introduced (e.g., dependence of the measure of damage on the mean pressure, use of a non-straight

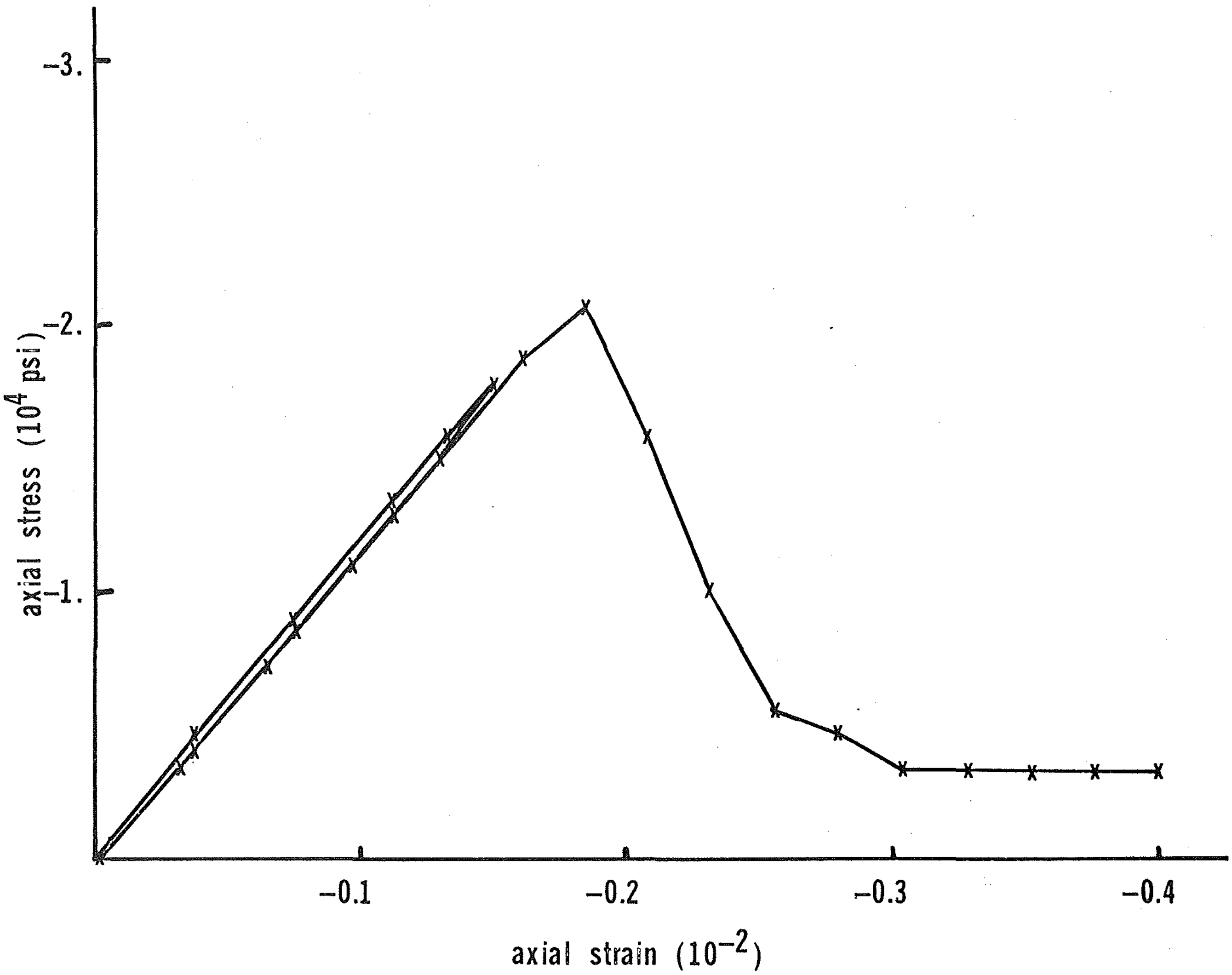


Figure 21 Model Predictions for Pre – peak Strength Unloading and Reloading of Tennessee Marble II.

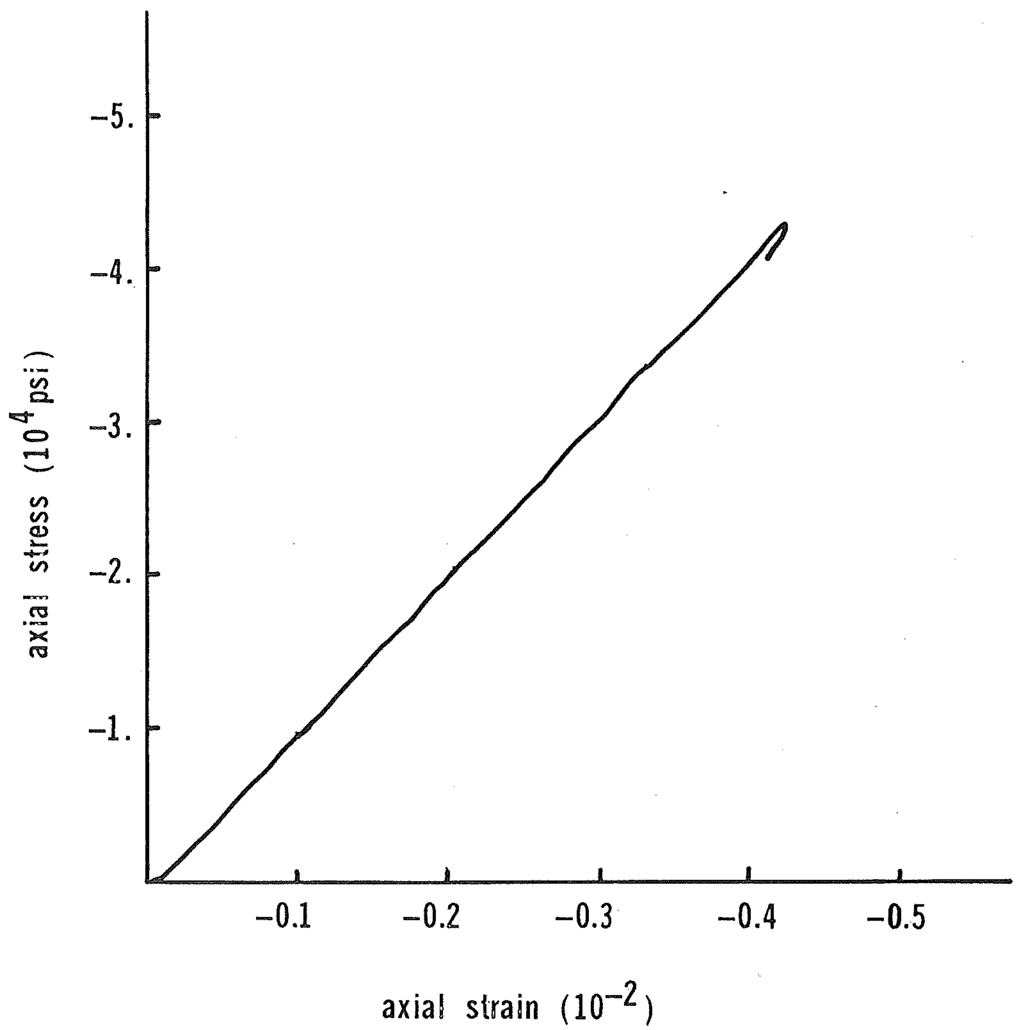
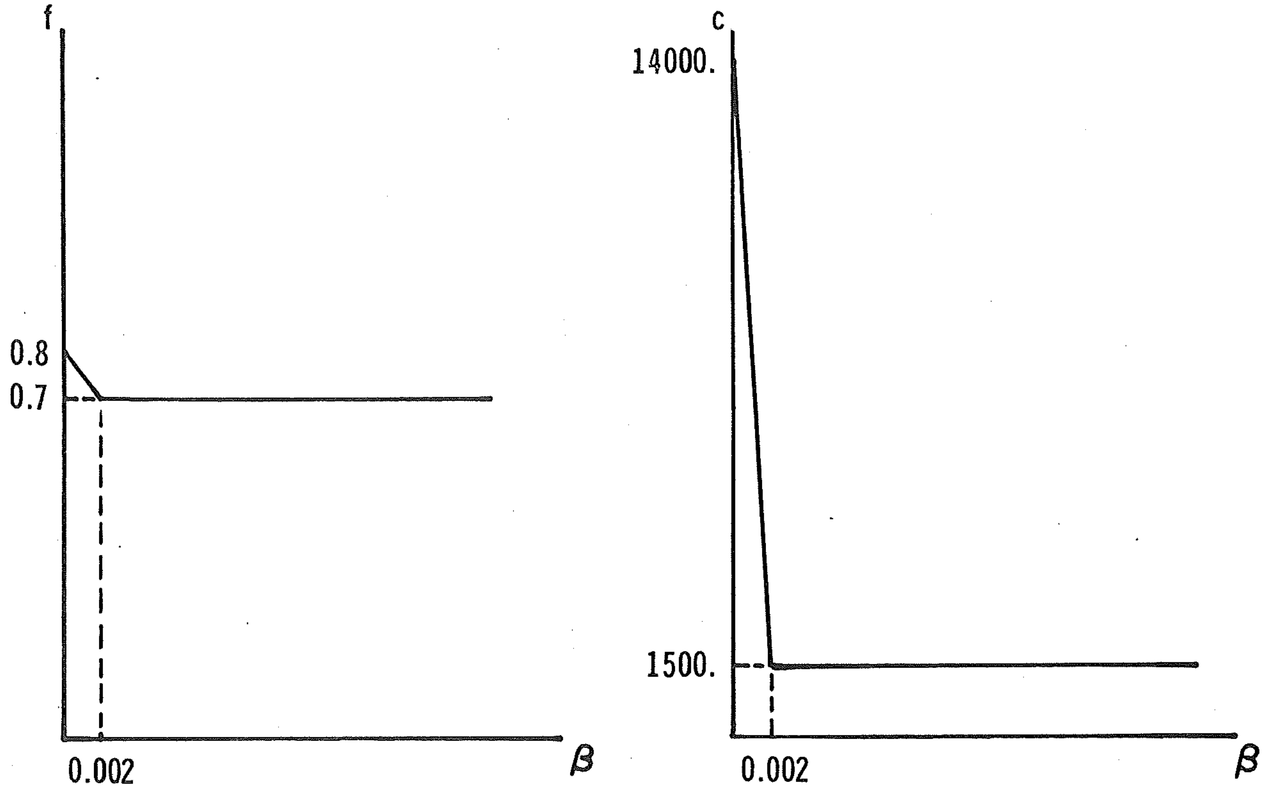
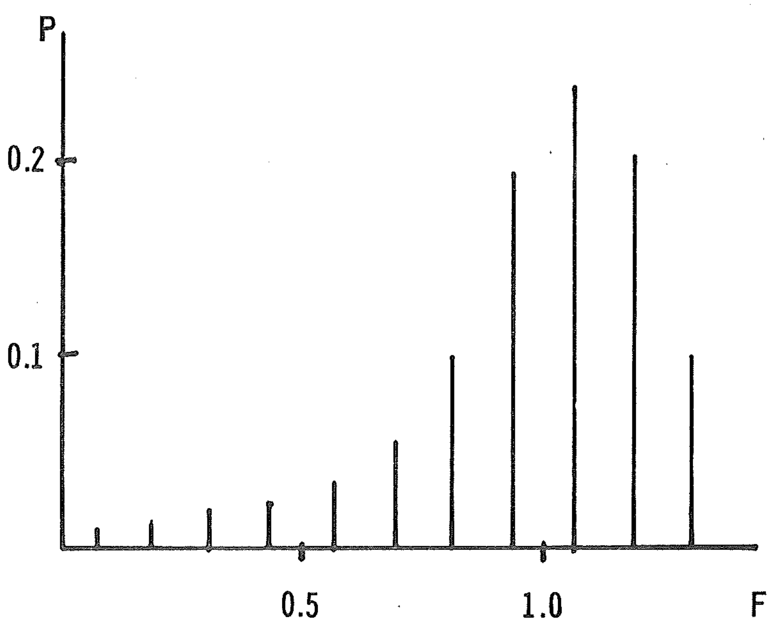


Figure 22 Experimental Results Taken from Wawersik – 70 for a Uniaxial Test of Solenhofen Limestone.





NONP = 0
D_r = 100.
E = 10325000.
ν = 0.27
σ_t = 2000.

Figure 23 Model Parameters Used in Study of Solenhofen Limestone .

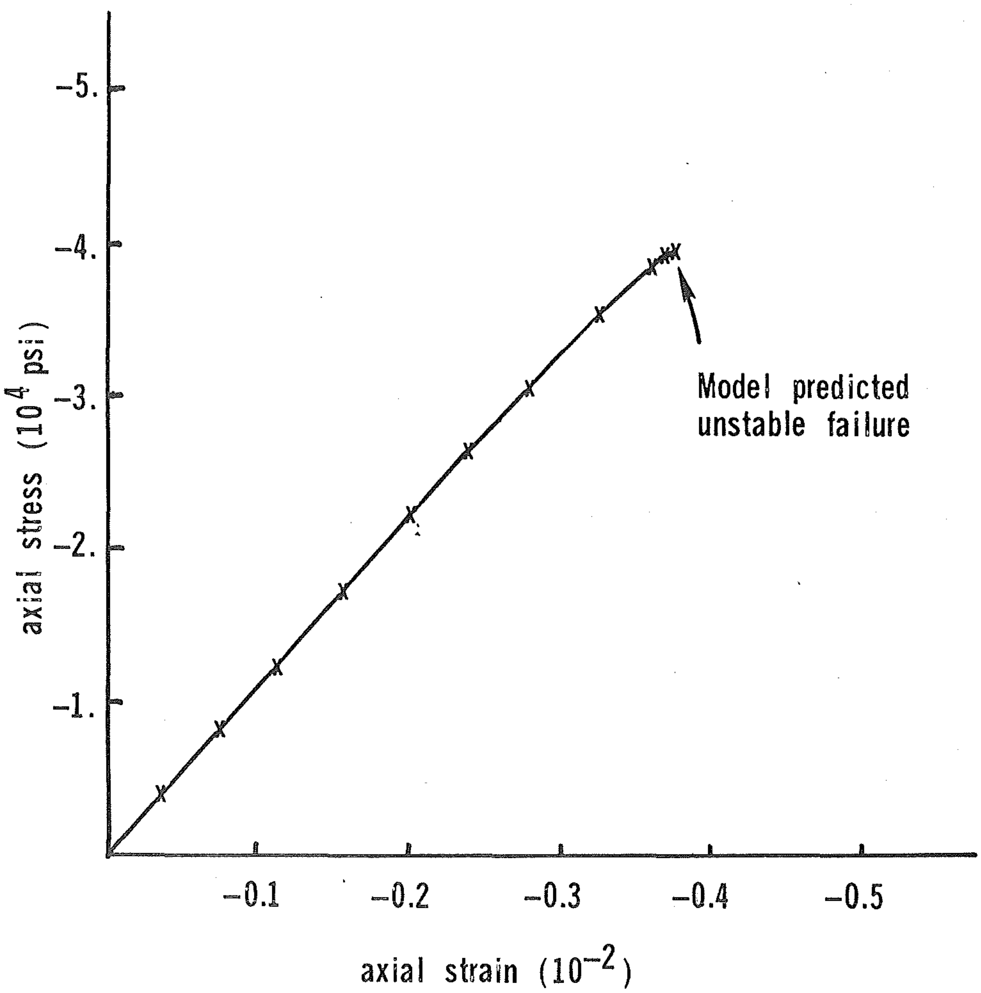


Figure 24 Predicted Stress – strain Curve (Zero Confining Pressure – State of Plane Strain) for Solenhofen Limestone.

failure envelope, etc.).

A really convincing substantiation of the model will, however, only be possible once the model is incorporated into a finite element program, and used for the analysis of actual rock structures for which carefully taken measurements are available.

Finally, it is of interest to consider the "structural peak strength envelope" predicted by the model, i.e., the envelope of the Mohr's circles for peak strength conditions. Plots of several such Mohr's circles are given in Figure 25; an expanded view of the area near the origin is given in Figure 26*. Also shown in these figures is a plot of the average (taking into account the concept of the integrity factor, e.g., see equ. (8)) failure envelope (see equs. (14) and (15)) used at the particle level. Except for the tensile stress region the two are nearly identical; thus, indicating that a measurement of one can be used to predict the other. Thus, if the peak strength envelope is parabolic one would expect the macroscopic failure envelope to be parabolic, etc.

E. DETERMINATION OF MODEL PARAMETERS:

The final task of the characterization procedure is the calibration of the model, i.e., for the rock mass of interest, the determination of numerical values for the several parameters which describe the representative volume, i.e., $c(\beta)$, $f(\beta)$, σ_t, etc., (see Section III-C-1).

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* These plots are based upon the assumption that at peak strength the stress normal to the plane is the intermediate principal stress; an assumption whose implications are not immediately obvious.

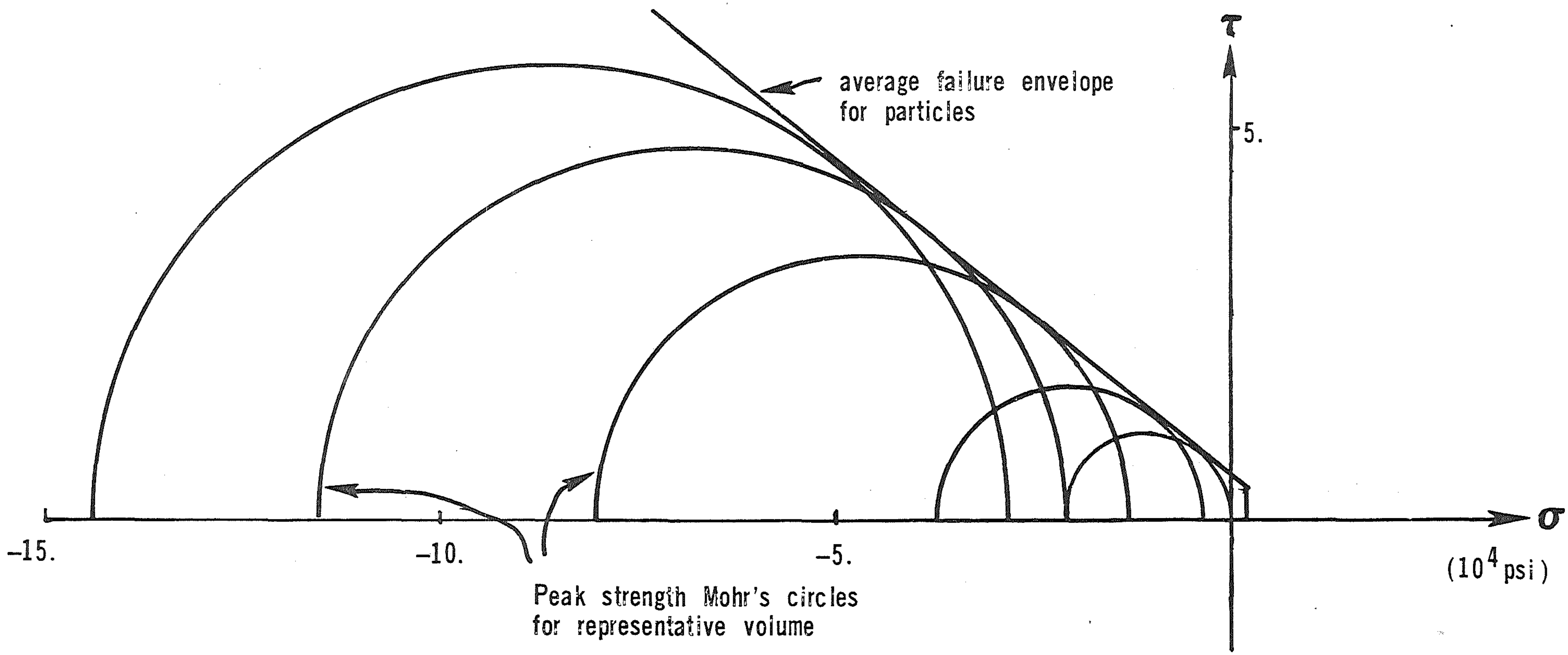

Figure 25 Comparison of "Structural" and Average "Macroscopic" Failure Envelopes.

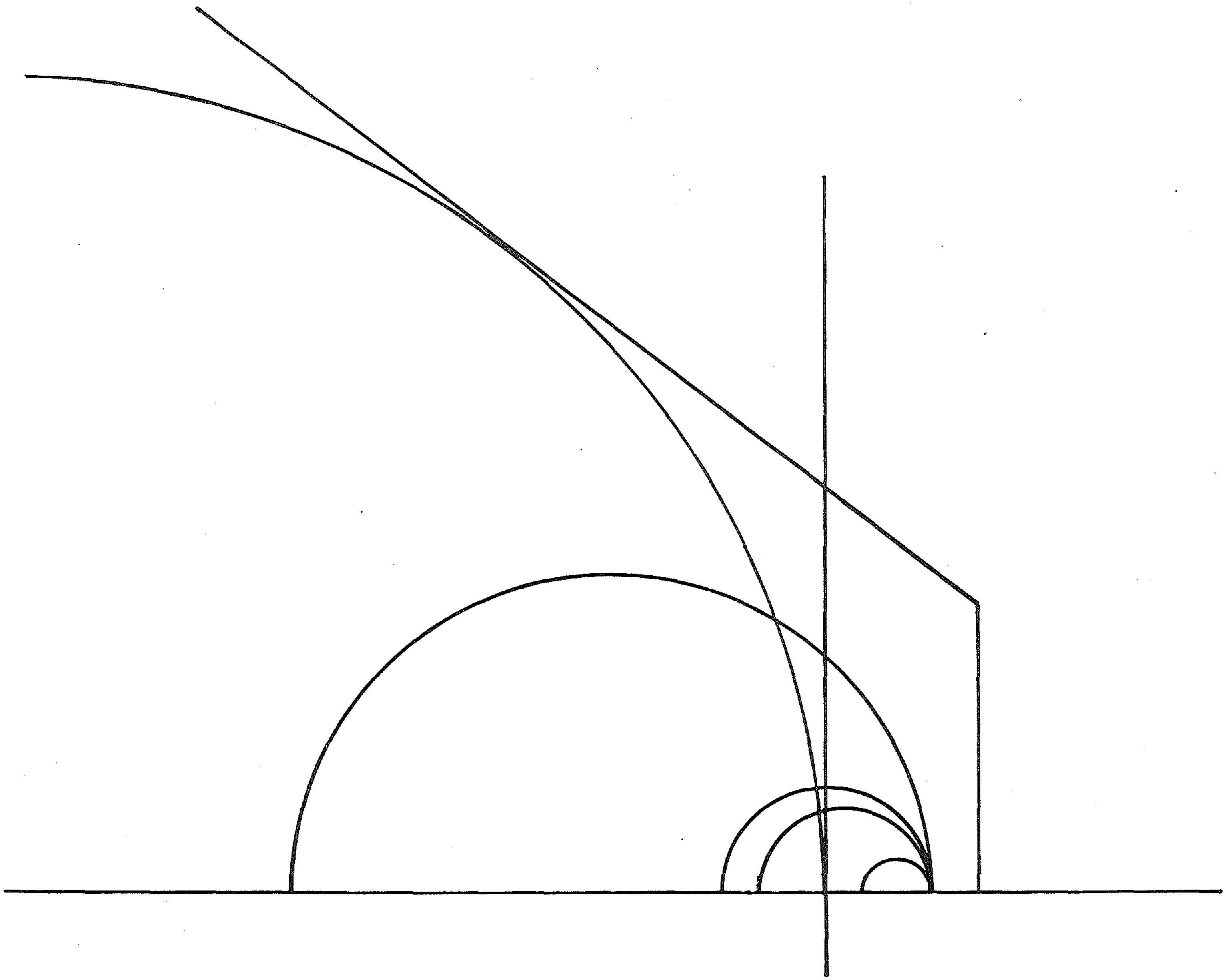

Figure 26 Details of Figure 25 for Small Stress Values.

It is envisioned that this process will consist of three steps:

a) The determination of the basic macroscopic rock properties, such as cohesion (see Figure 10), friction, variability (see Figure 4), etc., from fundamental observations of rock behavior in simple laboratory tests.

b) The use of the properties, determined in the previous step, for the prediction of the behavior of laboratory samples subjected to complicated stress and strain histories. These predictions would be compared with laboratory measurements, and the parameters determined in the previous step modified* in order to improve the "fit" of the data**.

c) The use of the properties, determined from the above two steps, in the prediction of the behavior of large scale rock structures for which experimental results are available. If necessary, the parameters would again be revised in order to yield the best comparison of analytical and experimental results. It is expected that the histogram of the integrity factor will be the quantity that will require

---

* There is currently considerable effort being expended on the development of systematic means for making such revisions, e.g., see Collins-74.

** The concept of revising the "fundamental" properties determined in step "a" in light of the results of steps "b" and "c" is justified by the following consideration: While the basic parameters, which describe the fundamental aspects of rock behavior (e.g., cohesion), are introduced from a consideration of the mechanics of rock behavior, they may ultimately be treated as merely arbitrary parameters which describe a proposed model. As parameters, which describe a proposed structural characterization model, they should ultimately be selected to yield the best agreement with experimental observation of rock behavior at the structural level.

the greatest revision.

To date, the only effort that has been expended on this phase of the research is the gathering together of some published data concerning values of the basic parameters, see Appendix D. Upon completion of the three-dimensional model, the major research effort will be concentrated on this calibration task.

IV RECOMMENDATIONS FOR FUTURE WORK

The research and development which is required to complete the overall project (see Section II) may be categorized into five main steps:

a) The improvement of the plane strain model.

b) The incorporation of the plane strain model into an existing two-dimensional finite element program, and the evaluation of the overall effectiveness of the resulting program.

c) The extension of the model to the general three-dimensional case, and the incorporation of the results into an existing three-dimensional finite element program.

d) The development of a systematic procedure for determining the model parameters.

e) The substantiation of the ability of the model to represent the important structural characteristics of rock masses. This verification will include the comparison of analyses of large scale rock structures with available experimental measurements.

The execution of these five steps will, of necessity, require a certain amount of iteration, e.g., discrepancies between experimental and analytical results observed in step "e" may lead to a revision of the basic model (step "a"), etc.

At this point in the project, it is possible to list those items that need to be covered in the initial step of the continued research effort, i.e.:

a) The modification of subroutine PROP to include the effects of residual stresses, a maximum of three planes of weakness (instead of two as is now the case), a more realistic description of the behavior of joints (e.g., including dilatation effects, the initial state of the joint, etc., Goodman-72), a more accurate failure criterion (e.g., a parabolic Mohr's envelope), the effects of pore water pressure, and print statements to indicate, at various stages of the analysis, the degree of rock damage.

b) The development of a more rational definition for the measure of damage β (see Section III-B-3); of a better understanding of the process which governs fragmentation; and, possibly, a more accurate description of the rubble phase.

c) A study to determine the importance of accounting for rate effects, temperature effects, size effects, variability of the stiffness parameters (E and ν for isotropic rock) and the coefficient of friction, and the nonlinear effects caused by closing of pores at very low stress levels.

d) The determination of the effect of the number of intact particles (NODIS) on the economy and accuracy of the model.

e) A study of the desirability and feasibility of revising the basic assumptions underlying the use of equ. (3) and of Table 1.

V CONCLUSIONS

The basic conclusions drawn from this research project are:

a) The development of a comprehensive characterization of the structural properties of rock masses, in terms of a representative volume, appears to be feasible.

b) The representative volume described in this report appears to be capable of representing, in a qualitative fashion, the general characteristics of rock behavior for a state of plane strain. However, several relatively straightforward revisions are needed in order to improve the quantitative aspects of the representation.

c) The proposed characterization for rock masses is significantly more advanced than any other currently available model.

d) The process of incorporating the subroutine, that has been developed for evaluation of the proposed model, into existing finite element programs appears to be relatively a simple task.

e) A general three-dimensional model could be developed as an extension of the plane strain model presented in this report. Conceptually this development is relatively straightforward, although the algebraic and numerical difficulties will be substantial.

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VII APPENDICES

A. DETERMINATION OF FAILURE PLANE ORIENTATION

In this appendix the procedure for determining the minimum positive value of the quantity k defined by equ. (26) is established. Because this task is related to the problem of finding principal stresses, it is relatively simple to demonstrate that there is always one or more such minima, except for certain very specialized cases, e.g., case of a hydrostatic stress state in an isotropic rock.

Ignoring for a moment the fact that theoretically k should not be negative, a minimum is sought by setting the derivative of equ. (26) equal to zero, i.e.,

$$k(\theta) = \frac{u(\theta)}{v(\theta)} \tag{A-1}$$

Set

$$\frac{dk}{d\theta} = 0 = \frac{u'v - uv'}{v^2} \tag{A-2}$$

Where $u' \equiv \frac{du}{d\theta}$

Let $g(\theta) = u'v - uv'$ (A-3)

Thus $g(\theta) = 0$ (A-4)

The expression for g is a strongly nonlinear function of θ and thus is solved by iteration. In order to ensure that this process will converge to the correct result, a relatively good initial approximation (θ_o) is first determined.

Supposing for a moment that the <u>total</u> stress history is proportional (as noted in Section II-B, this is seldom true), the stress increment can then be expressed in the following simple form:

$$(\delta = \frac{\Delta\sigma_{x_N}}{\sigma_{x_N}} = \frac{\Delta\sigma_{y_N}}{\sigma_{y_N}} = \frac{\Delta\tau_{xy_N}}{\tau_{xy_N}}).$$

$$[\Delta\sigma]_N = \delta\ [\sigma]_N$$

Equ. (25) then yields:

$$v = (c - u)\,\delta$$

and $$v' = (c' - u')\,\delta$$

For this special case equ. (A-3) yields:

$$u'c - u\,c' = 0$$

If $c' = 0$ (isotropic material) the above equation yields:

$$u' = 0$$

Utilizing equ. (24) gives:

$$(f\,\frac{\sigma_x-\sigma_y}{2} + \eta\,\tau_{xy})\sin 2\theta_o - (f\,\tau_{xy} - \eta\,\frac{\sigma_x-\sigma_y}{2})\cos 2\theta_o = 0$$

or

$$\tan 2\theta_o = \frac{f\,\tau_{xy} - \eta\,\frac{\sigma_x-\sigma_y}{2}}{f\,\frac{\sigma_x-\sigma_y}{2} + \eta\,\tau_{xy}}$$

The above expression has two roots of interest, i.e.,

$$\theta_o' = \frac{1}{2} \, \mathrm{Tan}^{-1} \left\{ \frac{f \, \tau_{xy} - \eta \, \frac{\sigma_x - \sigma_y}{2}}{f \, \frac{\sigma_x - \sigma_y}{2} + \eta \, \tau_{xy}} \right\}$$

and $$\theta_o'' = \theta_o' + \pi/2$$

The values of k, corresponding to each of these two angles, are calculated (the special considerations involved in this determination are described later in this appendix) and the one yielding the lesser value is used to define θ_o.

The Newton method (e.g., see Carnahan-69) is used to solve equ. (A-4); θ_o is used as the initial estimate. Each iteration of the Newton method requires the following calculation:

$$\theta_{i+1} = \theta_i - \frac{g(\theta_i)}{g'(\theta_i)}$$

Iteration is continued until the change between two successive iterations is less than approximately one degree (because of the inherent uncertainties involved in the characterization of rock, a more accurate determination is not justified).

When using this solution method, care must be taken to avoid the neighborhoods of angles for which $g'(\theta) = 0$ (i.e., a maximum or minimum point for g). If $g'(\theta) = 0$ or if it is sufficiently small to make $|\theta_{i+1} - \theta_i| \geq 2.0$ rad., then in order to move from this neighborhood and continue the iteration in an orderly manner, θ_i is arbitrarily changed by ± 0.4 rad. The plus sign is used if it gives the larger

The above expression has two roots of interest, i.e.,

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and $$\theta_o'' = \theta_o' + \pi/2$$

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Iteration is continued until the change between two successive iterations is less than approximately one degree (because of the inherent uncertainties involved in the characterization of rock, a more accurate determination is not justified).

When using this solution method, care must be taken to avoid the neighborhoods of angles for which $g'(\theta) = 0$ (i.e., a maximum or minimum point for g). If $g'(\theta) = 0$ or if it is sufficiently small to make $|\theta_{i+1} - \theta_i| \geq 2.0$ rad., then in order to move from this neighborhood and continue the iteration in an orderly manner, θ_i is arbitrarily changed by $\pm$ 0.4 rad. The plus sign is used if it gives the larger value of $v(\theta)$ (i.e., denominator of equ. (A-1)) and vice versa. If such a situation arises more than three times during the iteration process, it is taken as an indication that a minimum does not exist, and therefore all angles are equally critical; at this point the iteration process is stopped and the last value of θ_i is arbitrarily used.

Once the critical value of θ is determined it is then necessary to calculate the corresponding value of k. Because of the condition that $k \geq 0$ and because the limitations placed upon the permissible particle behavior modes (Table 1) may have artificially prevented a failure from occuring in the previous increment, this calculation involves some additional considerations beyond the straightforward use of equ. (A-1).

For non-failure conditions (i.e., equ. (23) not satisfied) the "reserve strength", R_s, against failure can be measured by the expression:

$$R_s = u - kv$$

At the beginning of the increment $k = 0$ and u is therefore the reserve strength (it is thus also the reserve strength at the end of the previous increment). With the relationships of u and v to R_s in mind, the values of $u(\theta)$ and $v(\theta)$ are calculated and the following interpretations placed upon their signs:

$u > 0.0$ - no tendency for failure at end of previous increment

$u \leq 0.0$ - failure prevented in previous increment by limitation of Table 1

$v > 0.0$ - tendency for particle failure during this increment

$v \leq 0.0$ - no tendency for particle failure during this increment.

The above considerations lead to the following specifications for the value of k:

$v < 0.0$, set $k > 1.0$ (indicates no failure during this increment)

$v \geq 0.0$ and $u \leq 0.0$, set $k = 0$ (failure at beginning of increment)

$v > 0.0$ and $u \geq 0.0$, set $k = u/v$

To this point, the question of the value of the parameter η which appears in equs. (18) and (21) has been ignored. Three values must be considered, i.e., $\eta = 0$ (tension failure) and $\eta = \pm 1$ (shear failure). The requirement that both $\eta = \pm 1$ be considered is apparent from a consideration of Figure A-1.

The computational steps outlined in this appendix are performed three times, i.e., for $\eta = 0, +1, -1$; the corresponding values of k are compared and the smallest selected as the critical value (the corresponding angle is the critical angle).

At this point, some consideration needs to be given to the fact that for certain situations there is a non-uniqueness involved in determining the critical shear failure plane (e.g., see Figure A-1; theoretically, fractures on the planes defined by $\eta = \pm 1$ are equally likely). Practically, this problem is not apparent because the numerical scheme used to determine the values of k always indicates one value higher than the other (due to round-off-error and incomplete convergence), i.e., an arbitrary selection is made of one of the orientations over the other. Because, for each of the particles, the

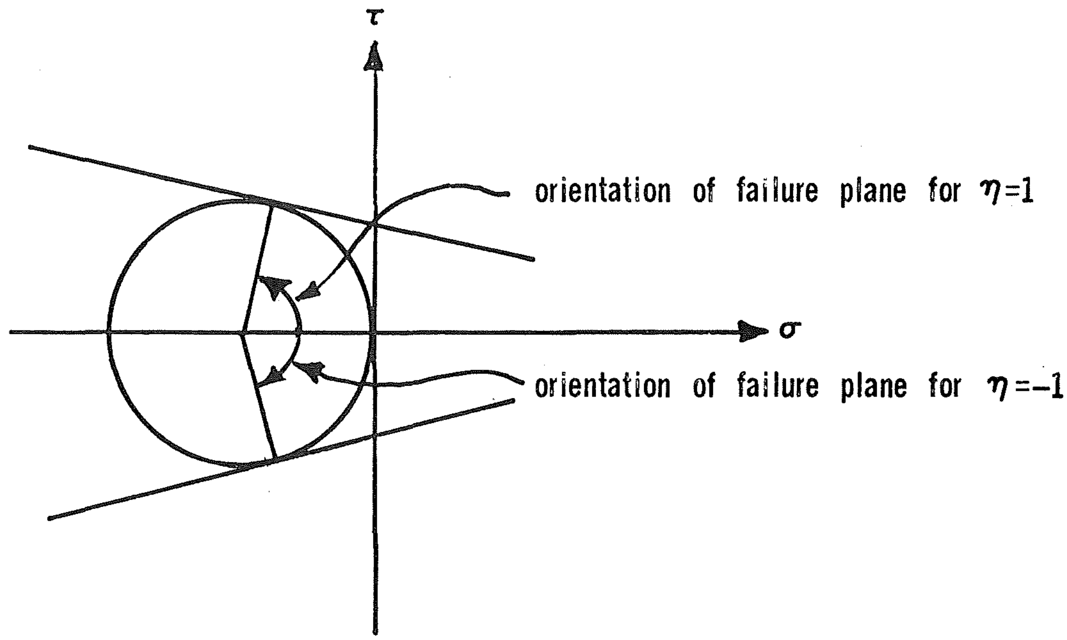

Figure A–1 Mohr Envelope for Isotropic Material; Failure in Uni–axial Compression.

sequence of operations are performed in the same order they all experience for a given increment the same bias in this selection process. This process may be viewed as an arbitrary selection for the first particle and a resulting influence of this first selection on those for the remaining particles. This process can be likened to what happens in the physical event. In the physical situation, the orientation of the first crack is influenced by arbitrary factors, such as initial imperfections, and the remainder by the bias introduced in the stress field due to the presence of the first crack.

```
      SUBROUTINE PROP(INTR,NELM,ISYMM,IC,DESP,CS,XLS)
C
C     SUBROUTINE TO CALCULATE INCREMENTAL STRUCTURAL PROPERTIES FOR
C     ROCK MASSES - BASED ON ANALYSIS OF REPRESENTATIVE VOLUME.
C     DEVELOPED BY L. R. HERRMANN, DEPT. OF CIVIL ENGINEERING,
C     UNIV. OF CALIF., DAVIS, 1974
C
      COMMON /B1/ ICNT,                      SPCF(1), SLIDM(20,3),
     1  DSLIDM(20,3), SEP(20,3), DSEP(20,3), SIGP(20,3), DSIGP(20,3),
     2  THA(20,3), NOFPA(20), NOFPD(20), DES(3), ES(3), NOF(3), FV(3,5),
     3  FS(3,5), NCC(3), CV(3,5), CSL(3,5), SIGTO(3), THS(3), POP(20),
     4  VAL(20), CE(3,3), NONP, NODIS, RAT, DS, FPAR, BLKMOD, DAMRAT,
     5  DFPAR
      COMMON /B2 / CEZ(3,3,4), SIGTOZ(3,4), FVZ(3,5,4), FSZ(3,5,4),
     1  CVZ(3,5,4), CSLZ(3,5,4), POPZ(20,4), VALZ(20,4), RATZ(4), DAMRTZ
     2  (4), THSZ(3,4), NONPZ(4), NODISZ(4), NCCZ(3,4), NOFZ(3,4)
      DIMENSION DESP(3),XLS(3),XLF(3),CS(3,3),CPA(3,3),CPB(3,3),NDU(2),
     1  P3(2),SF(3),DSP(3),CSV(3),FSV(3),SIGTS(3),SAV(3),XK(3),D(3),
     2  AB(3),AF(3),B(3),          H(3),XM(3),FPARST(3)
      NSTR = 596
  800 FORMAT(4I5,E10.3)
  801 FORMAT(2I5,2E10.3)
  802 FORMAT(8E10.2)
  900 FORMAT(5X,5HNONP=,I5,5X,6HNODIS=,I5,5X,'DR=',1PE10.3)
  901 FORMAT(//5X'STRENGTH PARAMETERS FOR PLANES OF WEAKNESS AND SOUND R
     1OCK'/8X,'SYSTEM =',I5//11X,'TENSILE STRENGTH =',1PE10.3,/11X,'ANGL
     2E =',1PE10.3//11X,'COHESION        DAMAGE')
  902 FORMAT(/,5X,'ELASTIC PROPERTIES'/20X,'E =',1PE10.3/8X,'POISSON RAT
     1IO =',1PE10.3)
  903 FORMAT(/20X,'VARIABILITY'/11X,'INTEGRITY FACTOR',4X,'PROPORTION')
  904 FORMAT(14X,1PE10.3,7X,1PE10.3)
  905 FORMAT(/,5X,'ANISOTROPIC ELASTIC PROPERTIES'/8X,'ANGLE PHI =',
     1  1PE10.3/16X,'R =',1PE10.3//,8X,'MATRIX OF ELASTIC CONSTANTS - C'
     2  / (11X,1P3E15.3))
  906 FORMAT(10X,1PE10.3,4X,1PE10.3)
  907 FORMAT(//,5X,'MATERIAL NUMBER =',I5,//)
  908 FORMAT(/11X,'FRICTION         DAMAGE')
      GO TO (10,120,190),INTR
C
C     READ MATERIAL PARAMETERS
C
   10 ICNT = 0
      READ(5,800) NOMAT
      DO 99 MNZ=1,NOMAT
      READ(5,800) MN,NONPZ(MN),NODISZ(MN),ITYP,DAMRTZ(MN)
      WRITE(6,907) MN
      WRITE(6,900) NONPZ(MN),NODISZ(MN),DAMRTZ(MN)
      IF (ITYP .NE. 0) GO TO 20
      READ(5,802) E,GNU
      RATZ(MN) = 1.0
      ANG = 0.0
      WRITE(6,902) E,GNU
      XMU=E/(1.0+GNU)
      XLAM=GNU*XMU/(1.0-2.0*GNU)
      CEZ(1,1,MN) = XLAM+XMU
      CEZ(1,2,MN) = XLAM
      CEZ(1,3,MN) = 0.0
      CEZ(2,2,MN) = CEZ(1,1,MN)
```

```
      CEZ(2,3,MN) = 0.0
      CEZ(3,3,MN) = XMU*0.5
      GO TO 25
   20 READ(5,802) ((CEZ(I,J,MN),J=I,3),I=1,3),ANG,RATZ(MN)
   25 DO 30 I = 2,3
      DO 30 J=1,I
   30 CEZ(I,J,MN) = CEZ(J,I,MN)
      IF(ITYP .EQ. 0) GO TO 50
      WRITE(6,905)ANG,RATZ(MN),((CEZ(I,J,MN),J=1,3),I=1,3)
   50 N=NCNPZ(MN)+1
      DO  60 I=1,N
      READ(5,801) NOFZ(I,MN),NCCZ(I,MN),SIGTOZ(I,MN),THSZ(I,MN)
      WRITE(6,901) I,SIGTOZ(I,MN),THSZ(I,MN)
      THSZ(I,MN) = THSZ(I,MN)*0.0349066
      M = NCCZ(I,MN)
      READ (5,802) (CVZ(I,J,MN),CSLZ(I,J,MN),J=1,M)
      WRITE(6,906) (CVZ(I,J,MN),CSLZ(I,J,MN),J=1,M)
      M=NOFZ(I,MN)
      READ (5,802) (FVZ(I,J,MN), FSZ(I,J,MN),J=1,M)
      WRITE(6,908)
   60 WRITE(6,906) (FVZ(I,J,MN), FSZ(I,J,MN),J=1,M)
      M = NCDISZ(MN)
      READ(5,802)  (POPZ(I,MN),VALZ(I,MN),I=1,M)
      WRITE(6,903)
      WRITE(6,904) (VALZ(I,MN),POPZ(I,MN),I=1,M)
C
C      TRANSFORM ORTHOTROPIC PROPERTIES TO X-Y AXES
C
      IF(ANG .EQ. 0.0) GO TO 99
      ANG = ANG*.0349066
      SA = SIN(ANG)
      CA = COS(ANG)
      CS(1,1) = 0.5*(1.0+CA)
      CS(1,2) = 0.5*(1.0-CA)
      CS(1,3) = -SA
      CS(2,1) = CS(1,2)
      CS(2,2) = CS(1,1)
      CS(2,3) =  SA
      CS(3,1) = 0.5*SA
      CS(3,2) =-0.5*SA
      CS(3,3) = CA
      DO 75 J=1,3
      DO 75 L=1,3
      DU = 0.0
      DO 73 K=1,3
   73 DU = DU+CEZ(J,K,MN)*CS(L,K)
   75 CPA(J,L) = DU
      DO 80 I = 1,3
      DO 80 L = 1,3
      DU = 0.0
      DO 77 J=1,3
   77 DU = DU+CS(I,J)*CPA(J,L)
   80 CEZ(I,L,MN) = DU
      DU =        0.5*(1.0-CA+(1.0+CA)*RATZ(MN))
      XMUL =      0.5*(1.0+CA+(1.0-CA)*RATZ(MN))
      M = NCCZ(I,MN)
      DO 85 J=1,M
   85 CVZ(N,J,MN) = CVZ(N,J,MN)*XMUL
      SIGTOZ(N,MN) = XMUL*SIGTOZ(N,MN)
      RATZ(MN) = DU/XMUL
```

```
   99 CONTINUE
      RETURN
C
C     INITIALIZE PROPERTY ARRAYS FOR EACH ELEMENT
C
  120 MN = IC
      NONP = NONPZ(MN)
      NODIS = NODISZ(MN)
      RAT = RATZ(MN)
      DAMRAT = DAMRTZ(MN)
      N = NONP+1
      DO 124 I=1,N
      SIGTO(I) = SIGTOZ(I,MN)
      NOF(I) = NOFZ(I,MN)
      NOC(I) = NOCZ(I,MN)
      THS(I) = THSZ(I,MN)
      DO 124 J=1,5
      FV(I,J) = FVZ(I,J,MN)
      FS(I,J) = FSZ(I,J,MN)
      CV(I,J) = CVZ(I,J,MN)
  124 CSL(I,J) = CSLZ(I,J,MN)
      DO 126 I=1,3
      DO 126 J=1,3
  126 CE(I,J) = CEZ(I,J,MN)
      DO 128 I=1,NODIS
      POP(I) = POPZ(I,MN)
  128 VAL(I) = VALZ(I,MN)
C
C     SIZE OF THE ORIGINAL FRAGMENTED MATERIAL PARTICLE ESTABLISHED
C
      FPAR = 1.0
      DO 129 I=1,NODIS
  129 FPAR = FPAR-POP(I)
      IF(FPAR .LT. 0.0) FPAR=0.0
      DFPAR = 0.0
      XLAM = CE(1,2)
      XMU = 0.5*(CE(1,1)+CE(2,2))-XLAM
      BLKMOD = XLAM+XMU/3.0
C
C     INITIALIZE SAMPLE STRAIN STATE AND PARTICLE STATE STORAGE
C
      DO 132 I = 1,3
      DES(I) = 0.0
  132 ES(I) = 0.0
      DS     =      FS(1,2)
      DO 135 I=1,NODIS
      NOFFD(I) = NONP
      DO 134 J=1,2
  134 THA(I,J) = THS(J)
      DO 135 J=1,3
      SIGP(I,J) = 0.0
      DSIGP(I,J)=0.0
      SLICM(I,J)= 0.0
      DSLICM(I,J)=0.0
      DSEP(I,J)=0.0
  135 SEP(I,J)=-1.0
      IF(NELM .EQ. 1) RETURN
      ICNT = ICNT+1
      WRITE(1) (SPCF(I),I=1,NSTR)
      RETURN
```

```
C
C     CALCULATION OF INCREMENTAL PROPERTIES
C
  190 IF(NELM .EQ. 1) GO TO 195
      IF(ICNT .LT. NELM) GO TO 193
      ICNT = 0
      REWIND 1
  193 ICNT = ICNT+1
      READ(1) (SPCP(I),I=1,NSTR)
      BACKSPACE 1
  195 IF(IC .GT. 1) GO TO 250
C
C     UPDATE PARTICLE STATE - BEGINING OF NEW INCREMENT
C
  200 DO 210 I=1,NODIS
      NOFP = NOFPD(I)
      IF(NOFP .EQ. 0) GO TO 206
      DO 205 J=1,NOFP
      SLIDM(I,J)=SLIDM(I,J)+DSLIDM(I,J)
  205 SEP(I,J)=DSEP(I,J)+ SEP(I,J)
  206 NOFPA(I) = NOFP
      DO 207 K=1,3
  207 SIGP(I,K)= SIGP(I,K)+DSIGP(I,K)
  210 CONTINUE
      DO 220 K = 1,3
  220 ES(K) = ES(K) + DES(K)
      FPAR = FPAR+DFPAR
C
C     CALCULATION OF INCREMENTAL PROPERTIES- FOR A GIVEN ITERATION
C
  250 DFPAR = 0.0
      VOLB = ES(1)+ES(2)
      VOLF = VOLB+DESP(1)+DESP(2)
      BLKMDB = BLKMOD
      IF(VOLB .GE. 0.0) BLKMDB=0.0
      FPARST(1) = BLKMDB*VOLB
      FPARST(2) = FPARST(1)
      FPARST(3) = 0.0
      DU = 0.0
      IF(VOLB .GE. 0.0 .AND. VOLF .GE. 0.0) GO TO 257
      IF(VOLB .LE. 0.0) GO TO 252
      DU = ABS(VOLF/(VOLF-VOLB))
      GO TO 257
  252 IF(VOLF .LT. 0.0) GO TO 256
      DU = ABS(VOLB/(VOLF-VOLB))
      GO TO 257
  256 DU = 1.0
  257 BLKMDN = DU*BLKMOD
      DO 260 K=1,3
      DES(K) = DESP(K)
      XLS(K) = 0.0
      DO 260 L=1,3
  260 CS(K,L) = 0.0
      DS     =0.5*DS
      DO 700 I=1,NODIS
      NOFP = NOFPA(I)
      NOFPD(I) = NOFP
      DO 303 K=1,3
      DSLIDM(I,K) = 0.0
      DSEP(I,K) = 0.0
```

```
      THS(K) = THA(I,K)
  303 XLF(K) = 0.0
      ID=1
      PP=FCF(ID)
      XINIG=VAL(ID)
      N=0
      PB   =0.0
      IPB = 0
      P2 = 0.0
      NS = 0
C
C        DETERMINE NO. OF OPEN CRACKS IN PARTICLE
C
      IF(NCFP .EQ. 0) GO TO  309
      DO 304 J=1,NCFP
      IF(SEP(I,J) .LE. 0.0) GO TO 304
      N=N+1
      NDL(N)=J
  304 CONTINUE
      P1 = 0.0
      NS = N
      DSEFSV = 0.0
      SAV(1) = 0.0
      SAV(2) = 0.0
      SAV(3) = 0.0
      IF(N-1)309,314,342
C                               ELASTIC RESPONSE - NO OPEN CRACKS
  309 P1 = 1.0
      GO TO 360
C                               ELASTIC RESPONSE - ONE OPEN CRACK
  314 P2 =1.0
      J=NDL(1)
  317 SIGNN = 0.0
      SIGNS = 0.0
  318 TH=THS(J)
      CA=COS(TH)
      SA=SIN(TH)
      AB(1) = 0.5*(1.0+CA)
      AB(2) = 0.5*(1.0-CA)
      AB(3) = SA
      B(1) = -0.5*SA
      B(2) = -B(1)
      B(3) = CA
      DO 322  K=1,3
      DU1 = 0.0
      DU2 = 0.0
      DO 320 L=1,3
      DU1= CF(K,L)*AB(L)+DU1
  320 DU2= CF(K,L)* B(L)+DU2
      AF(K) = DU1
      XKI11 = 0.0
  322 D(K) = DU2
      XKI12 = 0.0
      XKI22 = 0.0
      DO 325 K=1,3
      XKI11 = XKI11 + AF(K)*AB(K)
      XKI12 = XKI12 + AB(K)* D(K)
  325 XKI22 = XKI22 +  B(K)* D(K)
      DU1 = 1.0/(XKI11*XKI22-XKI12*XKI12)
      DU2 = XKI11
```

```
      XKI11 = -XKI22*DU1
      XKI12 =  XKI12*DU1
      XKI22 = -DU2*DU1
      DO 330 K=1,3
      H(K) = -AF(K)*XKI11 -D(K)*XKI12
  330 XM(K) =-AF(K)*XKI12 -D(K)*XKI22
      DU1 = DSEPSV
      DO 340 K=1,3
      DO 335 L=1,3
  335 CPB(K,L) = CE(K,L)-AF(K)*H(L)-D(K)*XM(L)
      DU1      = DU1    +P2*H(K)*DES(K)
  340 XLP(K) = -(XKI11*AF(K)+D(K)*XKI12)*SIGNN
     1          -(XKI12*AF(K)+D(K)*XKI22)*SIGNS
      DSEP(I,J) = DU1    +XKI11*SIGNN+XKI12*SIGNS
      IF(IPB .EQ. 1) GO TO 700
C                           MODIFY RESPONSE IF CRACK CLOSES
      IF (SEP(I,J)+DSEP(I,J) .GE. 0.0) GO TO 360
      NS = NS-1
      P2P =-P2*(SEP(I,J)+DSEPSV)/(DSEP(I,J)-DSEPSV)
      P1 = P2-P2P
      P2 = P2P
      DSEP(I,J) =-SEP(I,J)
      GO TO 360
C                           ELASTIC RESPONSE - TWO OR MORE OPEN CRACKS
  342 P2 = 1.0
  343 J1 = NDU(1)
      TH1 = THS(J1)
      CA1 = COS(TH1)
      SA1 = SIN(TH1)
      DENO = 0.5*((1.0+CA1)*SAV(1)+(1.0-CA1)*SAV(2)+SA1*SAV(3))
      DESO = 0.5*((1.0-CA1)*SAV(1)+(1.0+CA1)*SAV(2)-SA1*SAV(3))
      DEN  = 0.5*((1.0+CA1)*DES(1)+(1.0-CA1)*DES(2)+SA1*DES(3))*P2
      DES1 = 0.5*((1.0-CA1)*DES(1)+(1.0+CA1)*DES(2)-SA1*DES(3))*P2
      J2 = NDU(2)
      DTH = THS(J2)-TH1
      CDTH = COS(DTH)
      DUN2 = 2.0*(DES1+DESO)/(1.0-CDTH)
      DUN1 = DEN+DENO-0.5*(1.0+CDTH)*DUN2
      IF(CDTH .LT. -.99754) GO TO 353
      SEPP=SEP(I,J1)+DSEP(I,J1)
      IF(SEPP     +DUN1 .GE. 0.0 .AND. SEP(I,J2)+DUN2 .GE. 0.0)GO TO 353
      DU = DEN+DENO+DES1+DESO
      IF(SEPP     +DUN1 .LT. SEP(I,J2)+DUN2) GO TO 348
      DUN2 = -SEP(I,J2) * .999999
      DUN1 = DU-DUN2
      IF(SEPP     +DUN1 .GT. 0.0) GO TO 353
      DUN1 = -SEPP      * .999999
      DUN2 = DU-DUN1
      GO TO 353
  348 DUN1 = -SEPP      * .999999
      DUN2 = DU-DUN1
      IF(SEP(I,J2)+DUN2 .GT. 0.0) GO TO 353
      DUN2 = -SEP(I,J2) * .999999
      DUN1 = DU-DUN2
  353 DSEP(I,J1) = DSEP(I,J1)  +DUN1
      DSEP(I,J2) = DSEP(I,J2)  +DUN2
      IF(IPB .EQ. 0) GO TO 354
      IF(SEP(I,J2)+DSEP(I,J2) .GE. 0.0)GO TO 700
      DEN=-DSEP(I,J2)-SEP(I,J2)
      DSEP(I,J2)=-SEP(I,J2)
```

```
      GO TO 700
  354 DO 355 JJ=1,2
      J=NCL(JJ)
      P3(J)= 1.0
      IF(SEP(I,J)+DSEP(I,J) .GE. 0.0) GO TO 355
      NS = NS-1
      P3(J)=-SEP(I,J)/DSEP(I,J)
  355 CONTINUE
      P2 = 0.0
      IF(NS .GE. 2) GO TO 360
C                             MODIFY RESPONSE IF ONE OR MORE CRACK CLOSE
      DL = P3(1)
      KL = 2
      JJ = 1
      IF(DL .LT. P3(2)) GO TO 357
      KL = 1
      DL = P3(2)
      JJ = 2
  357 J = NCU(KL)
      DSEPSV = DSEP(I,J)*DL
      L=NCL(JJ)
      DSEP(I,L) =-SEP(I,L)
      P2 = 1.0 - DL
      NS= 1
      GO TO 317
C
C                             ELASTIC PORTION OF INCREMENTAL PROPERTIES
C
  360 DO 365 K=1,3
      DO 365 L=1,3
  365 CPA(K,L) = P1*CE(K,L)+P2*CPH(K,L)
      IF(NS .NE. N) GO TO 700
      IF(NS .GT. 1) GO TO 700
C
C                          SLIDING OR CRACK FORMATION (OR REOPENING) RESPONSE
C
      DO 370 K = 1,3
      SP(K)=SIGP(I,K)
      DSP(K) = 0.0
      DO 370 L=1,3
  370 DSP(K) = DSP(K)+CPA(K,L)*DES(L)
      S1 = SP(1)
      S2 = SP(2)
      S12= SP(3)
      DS1 =DSP(1)
      DS2 =DSP(2)
      DS12=DSP(3)
      NCANG = NCFP
      IF(NCFP .EQ. 0) GO TO 510
C                             SEARCH FOR POSSIBLE FAILURE PLANE --
C                               CONSIDER PREVIOUSLY DEFINED PLANES
      DO 500 J=1,NCANG
      UN=SEP(I,J)
      XK(J)=1.0
      IF(UN .GT. 0.0)GO TO 500
      US=SLIDN(I,J)
      JJJ=J
      JJ=J
      IF(JJ .GT. NCNP+1) JJ=NCNP+1
      CALL INTP(JJ,C,F,SIGT,US,UN)
```

```
      IF(J .LE. NONP)GO TO 410
      C = C*XINTG
      SIGT=SIGT* XINTG
  410 CSV(J)=C
      FSV(J)=F
      SIGTS(J)=SIGT
      TH2 = THS(J)
      CTH = COS(TH2)
      STH = SIN(TH2)
      SAV(J) = 1.0
      SNS = 0.5*(S2-S1)*STH+S12*CTH
      TSNS=SNS+0.5*(DS2-DS1)*STH +DS12*CTH
      IF(TSNS .LT. 0.0) SAV(J)=-1.0
      S = 0.0
      CP = SIGT*F
      DO 430 K=1,2
      XKN = XKS
      U=CP-F*0.5*(S1+S2)-(F*0.5*(S1-S2)+S*S12)*CTH-(F*S12-S*0.5*(S1-S2))
     1       *STH
      V=F*(.5*(DS1+DS2)+(F*0.5*(DS1-DS2)+S*DS12)*CTH+(F*DS12-S*0.5*(DS1
     1      -DS2))*STH
      XKS = 10000.
      IF(U .GE. 0.0) GO TO 422
      IF(V .GE. 0.0) XKS = 0.0
      GO TO 428
  422 IF(V .LE. 0.0) GO TO 428
      XKS = U/V
  428 S = SAV(J)
  430 CP = C
      XK(J) = XKS
      IF(XKS .LT. XKN) GO TO 440
      SAV(J) = 0.0
      XK(J) = XKN
  440 S = SAV(J)
      P  = XK(J)
      IF(  P    .LT. .001) GO TO 540
  500 CONTINUE
      IF(NCFP .EQ. 3) GO TO 515
  510 NCANG = NCANG + 1
C                           LOOK FOR NEW FAILURE PLANE
      C = CV(NONP+1,1)*XINTG
      F = FV(NONP+1,1)
      SIGT = SIGTO(NONP+1)* XINTG
      CSV(NCANG) = C
      FSV(NCANG) = F
      SIGTS(NCANG) = SIGT
      CALL CRTANG(S1,S2,S12,DS1,DS2,DS12,F,C,SIGT,RAT,XKC ,S,TH2)
      SAV(NCANG) = S
      THS(NCANG) = TH2
      XK(NCANG) = XKC
  515 P  = 1.0
C                           DECIDE ON CRITICAL (IF ANY) FAILURE PLANE
      DO 520 J = 1,NCANG
      IF(XK(J) .GE. P ) GO TO 520
      JJJ = J
      S = SAV(J)
      P  = XK(J)
  520 CONTINUE
      IF(P  .GE. 1.0) GO TO 700
  540 J = JJJ
```

```
      JJ = J
      IF(JJ .GT. NONP+1)JJ = NONP+1
      TH2 = THS(J)
      CTH = COS(TH2)
      STH = SIN(TH2)
      SNN  =0.5*( S1+S2+P *(DS1+DS2) +(S1-S2+(DS1-DS2)*P )*CTH)+(S12+P *
     1      DS12)*STH
      IF(SNN   .GT. 0.01*SIGTO(JJ)) S=0.0
      PB = 1.0-P
      IF(J   .LE. NOFP) GO TO 543
      NOFP = NOFP+1
      NOFPD(I) = NOFP
      THA(I,NOFP) = TH2
  543 IF(S .EQ. 0.0) GO TO 600
C                          SLIDING RESPONSE
      CB = CSV(J)
      FB = FSV(J)
      IF(F .EQ. 0.0) CB= S*((S2-S1)*0.5*STH+S12*CTH)+FB*(0.5*(S1+S2)
     1      +0.5*(S1-S2)*CTH+S12*STH)
      DO 545 K=1,3
  545 D(K) = 0.5*(CPA(K,1)-CPA(K,2))*STH-CPA(K,3)*CTH
      AB(1) = -0.5*(FB*(1.0+CTH)-S*STH)
      AB(2) = -0.5*(FB*(1.0-CTH)+S*STH)
      AB(3) = -(FB*STH+S*CTH)
      UN = 0.0
      SMB = SLIDM(I,J)
      DO 565 KK=1,10
      SME = SMB+DS
      DSB=DS
      CALL INTP(JJ,CF,FF,SIGTF,SME,UN)
      IF(J .GT. NONP)CF=CF*XINTG
      AF(1) = -0.5*(FF*(1.0+CTH)-S*STH)
      AF(2) = -0.5*(FF*(1.0-CTH)+S*STH)
      AF(3) = -(FF*STH+S*CTH)
      DO 550 K=1,3
      DU = 0.0
      DO 548 L=1,3
  548 DU = DU+CPA(L,K)*AF(L)
  550 B(K) = DU
      XKI = 0.0
      DC = CF-CB
      DO 555 K=1,3
      DC = DC+(AF(K)-AB(K))*(SP(K)+P*DSP(K))
  555 XKI = XKI+AF(K)*D(K)
      XKI = 1.0/XKI
      DS = -DC
      DO 560 K=1,3
  560 DS = DS-B(K)*DES(K)*PB
      DS = DS*XKI
      DS = ABS( DS)
      DU=ABS((DS-DSB)/(DS+.0000000001))
      IF(DU .LT. .03) GO TO 575
      DS=DSB+1.0*(DS-DSB)
      IF(DS .LT. 0.0) DS=0.0
  565 CONTINUE
  575 DO 577 K=1,3
      XLP(K) = -DC*D(K)*XKI
      DO 577 L=1,3
  577 CPB(K,L) = CPA(K,L)-D(K)*B(L)*XKI
      DO 579 K=1,3
```

```
      DU = XLP(K)+SP(K)
      DO 578 L=1,3
  578 DU = DU+(CPA(K,L)*(1.0-PB)+CPB(K,L)*PB)*DES(L)
  579 DSIGP(I,K) = DU
      DU = 0.5*(DSIGP(I,1)+DSIGP(I,2)+(DSIGP(I,1)-DSIGP(I,2))*CTH)+
     .      DSIGP(I,3)*STH
      IF(DU .LE. CF/FF) GO TO 580
      S = 0.0
      GO TO 600
  580 DSLIDM(I,J)=DS
      DO 590 K=1,NCANG
      IF(SEP(I,K) .LE. 0.0) GO TO 590
      DU = 0.0
      DO 582 L=1,3
  582 DU = DU+H(L)*(SUS*B(L)+DES(L)*(1.0-P))
      DSEP(I,K) = DSEP(I,K)*P+DU
      IF(-DSEP(I,K) .GT. SEP(I,K))DSEP(I,K)=SEP(I,K)
      GO TO 595
  590 CONTINUE
  595 IF(SEP(I,J) .LT. -0.9) DSEP(I,J) = 1.0
      GO TO 700
C                          CRACKING RESPONSE
  600 DSEPSV = 0.0
      IF(SEP(I,J) .LT.-0.9)DSEPSV=1.0
      SIGNS = -(0.5*(S2-S1+P*(DS2-DS1))*STH+(S12+P*DS12)*CTH)
      SIGNN = -SNN
      NS = NS+1
      P2 = PB
      IPB = 1
      IF(NS .EQ. 1) GO TO 318
      N = 2
      DO 685 K=1,3
      SAV(K)=SP(K)+P*DSP(K)
      XLP(K)=-SAV(K)
      DO 685 L=1,3
  685 CPB(K,L) = CE(K,L)
      DSEP(I,J) = DSEPSV
      NCL(1) = J
      DO 690 K=1,NCANG
  690 IF(SEP(I,K) .GT. 0.0) NCL(2)=K
      J2=NCL(2)
      DSEP(I,J2)=DSEP(I,J2)*P
      DO 692 K=2,3
      DO 692 L=K,3
      DU = CPB(L,K-1)/CPB(K-1,K-1)
      SAV(L) = SAV(L)-SAV(K-1)*DU
      DO 692 M=K,3
  692 CPB(L,M) = CPB(L,M)-CPB(K-1,M)*DU
      SAV(3) = SAV(3)/CPB(3,3)
      DO 694 K=1,2
      DO 693 L=1,K
  693 SAV(3-K) = SAV(3-K)-SAV(4-L)*CPB(3-K,4-L)
  694 SAV(3-K) = SAV(3-K)/CPB(3-K,3-K)
      DO 697 K=1,3
      DO 697 L=1,3
  697 CPB(K,L) = 0.0
      GO TO 343
C
C     ACCOUNT FOR LOSS OF MATERIAL FROM PARTICLE BY FRAGMENTATION
C
```

```
  700 IF(DAMRAT .EQ. 0.0) GO TO 725
      FPARIB = 0.0
      FPARIF = 0.0
      DO 710 K=1,3
      FPARIB = FPARIB+SLIDM(I,K)
      FPARIF = FPARIF+SLIDM(I,K)+DSLIDM(I,K)
      IF(SEP(I,K) .GT. -1.0) FPARIB=FPARIB+.05/DAMRAT
  710 IF(SEP(I,K)+DSEP(I,K) .GT. -1.0)FPARIF=FPARIF+.05/DAMRAT
      PPB=PP*(EXP(-DAMRAT*FPARIB)*0.05+0.95)
      PP =PP*(EXP(-DAMRAT*FPARIF)*0.05+0.95)
      DPP = PPB-PP
      DFPAR = DFPAR+DPP
      DO 720 K=1,3
  720 XLS(K) = XLS(K)-DPP*(SIGP(I,K)-FPARST(K))
C
C   FORMATION OF COMPOSITE ELASTIC-SLIDING-CRACKING RESPONSE OF PARTICLE
C
  725 PA = 1.0-PB
      DO 740 K=1,3
      XLS(K) = XLS(K)+PP*XLP(K)
      DO 740 L=1,3
      CPA(K,L) = CPA(K,L)*PA+CPB(K,L)*PB
C                          COMBINING OF PARTICLE BEHAVIOR TO OBTAIN
C                               COMPOSITE PROPERTIES OF SAMPLE
  740 CS(K,L) = CS(K,L)+CPA(K,L)*PP
      DO 745 K=1,3
      DU = XLP(K)
      DO 743 L=1,3
  743 DU = DU+CPA(K,L)*DES(L)
  745 DSIGP(I,K) = DU
  760 CONTINUE
C
C     ACCOUNT FOR FRAGMENTED PARTICLE
C
      DU = (FPAR+DFPAR)*BLKMDN
      CS(1,1) = CS(1,1)+DU
      CS(1,2) = CS(1,2)+DU
      CS(2,1) = CS(2,1)+DU
      CS(2,2) = CS(2,2)+DU
C                          MAKE PROPERTIES SYMMETRICAL IF REQUIRED
      IF(ISYMN .EQ. 0) GO TO 780
      DO 775 I=1,3
      DO 775 J=1,3
      CPA(I,J) = CS(I,J)
  775 XLS(I) = XLS(I)+0.5*(CS(I,J)-CS(J,I))*DES(J)
      DO 777 I=1,3
      DO 777 J=1,3
  777 CS(I,J) = 0.5*(CPA(I,J)+CPA(J,I))
  780 IF(NELM .NE. 1) WRITE(1)(SPCP(I),I=1,NSTR)
      RETURN
      END
```

```
      SUBROUTINE INTP(J,C,F,SIGT,US,UN)
C
C     SUBROUTINE TO INTERPOLATE STRENGTH PARAMETERS
C
      COMMON  /B1/  ICNT,                    SPOP(1), SLIDM(20,3),
     1 DSLIDM(20,3), SEP(20,3), DSEP(20,3), SIGP(20,3), DSIGP(20,3),
     2 THA(20,3), NOFPA(20), NOFPD(20), DES(3), ES(3), NOF(3), FV(3,5),
     3 FS(3,5), NOC(3), CV(3,5), CS (3,5), SIGTO(3), THS(3), POP(20),
     4 VAL(20), CE(3,3), NONP, NODIS, RAT, DS, FPAR, BLKMOD, DAMRAT,
     5 CPPAR
      C = CV(J,1)
      SIGT = SIGTO(J)
      F = FV(J,1)
      IF(UN  .LT. -0.9) RETURN
      SIGT = 0.0
      NL = NOC(J)-1
      DO 300 L = 1,NL
      IF(US .GT. CS(J,L+1)) GO TO 300
      C = CV(J,L)+(CV(J,L+1)-CV(J,L))*(US-CS(J,L))/(CS(J,L+1)-CS(J,L))
      GO TO 350
  300 CONTINUE
      C = CV(J,NL+1)
  350 NL = NOF(J) -1
      DO 400 L=1,NL
      IF(US .GT. FS(J,L+1)) GO TO 400
      F = FV(J,L)+(FV(J,L+1)-FV(J,L))*(US-FS(J,L))/(FS(J,L+1)-FS(J,L))
      RETURN
  400 CONTINUE
      F = FV(J,NL+1)
      RETURN
      END
```

##

```
      SUBROUTINE CRTANG(SX,SY,SXY,DSX,DSY,DSXY,F,CV,ST,RAT, XKC,SC,THC)
C
C     SUBROUTINE TO CALCULATE MOST LIKELY FAILURE PLANE
C
      DIMENSION SA(3),TH(3),XKS(3),XK(2)
      SA(1) =0.0
      SA(2)= 1.0
      SA(3)=-1.0
      D1 = 0.5*(SX-SY)
      D2 = 0.5*(SX+SY)
      D3 = 0.5*(DSX-DSY)
      D4 = 0.5*(DSX+DSY)
C             LOOP TO CONSIDER POSSIBLE TENSION FAILURE AND
C             TWO POSSIBLE SHEAR FAILURE
      DO 300 J=1,3
      XKS(J)=10000.
      S=SA(J)
      X1 = F*D1+S*SXY
      DX1=F*D3+S*DSXY
      X2=F*SXY-S*D1
      DX2=F*DSXY-S*D3
C             INITIAL ESTIMATE
      TH2=ATAN2(X2,X1)-3.14159266
```

```
      DO 110 K=1,2
      TH2=TH2+3.14159266
      CTH=COS(TH2)
      STH=SIN(TH2)
      C=CV
      IF(J .EQ. 1)C =ST*F
      C = C*0.5*(1.0+CTH+(1.0-CTH)*RAT)
      U=C-F*D2             -X1*CTH-X2*STH
      V=F*D4               +DX1*CTH+DX2*STH
      XK(K)=10000.
      IF(U .GE. 0.0)GO TO 100
      IF(V .GE. 0.0)XK(K)=0.0
      GO TO 110
  100 IF(V .LE. 0.0) GO TO 110
      XK(K)=U/V
  110 CONTINUE
      IF( XK(1)  .LT. XK(2) )TH2=TH2-3.14159266
      KONT=0
C
C     SEARCH FOR ROOT BY MEANS OF NEWTONS METHOD
C
      DO 200 I=1,10
      CTH=COS(TH2)
      STH=SIN(TH2)
      C=CV
      IF(J .EQ. 1)C =ST*F
      CP = C*(RAT-1.0)*STH
      CPP = 2.0*C*(RAT-1.0)*CTH
      C = C*0.5*(1.0+CTH+(1.0-CTH)*RAT)
      U=C-F*D2             -X1*CTH -X2*STH
      V= F*D4               +DX1*CTH+DX2*STH
      UP=CP+2.0*(X1*STH-X2*CTH)
      VP=-2.0*(DX1*STH-DX2*CTH)
      UPP=CPP+4.0*(X1*CTH+X2*STH)
      VPP=-4.0*(DX1*CTH+DX2*STH)
      DU=TH2
      DU1=UPP*V-U*VPP
C                 AVOID MAX. OR MIN. POINT
      IF(DU1 .NE. 0.0)GO TO 130
  125 IF(KONT .EQ. 3)GO TO 290
      KONT=1+KONT
      DU2=0.4
      VF=F*D4+DX1*COS(TH2+DU2)+DX2*SIN(TH2+DU2)
      VB=F*D4+DX1*COS(TH2-DU2)+DX2*SIN(TH2-DU2)
      IF(VF .LT. VB)DU2=-DU2
      TH2=TH2+DU2
      GO TO 200
  130 DTH=  -(UP*V-U*VP)/DU1
      IF(ABS(DTH) .GT. 2.0)GO TO 125
      TH2=DU+DTH
      DU=ABS(TH2  -DU)
      IF(DU .LT. .035) GO TO 290
  200 CONTINUE
  290 TH(J)=TH2
      IF(U .GE. 0.0)GO TO 298
      IF(V .GE. 0.0)XKS(J)=0.0
      GO TO 300
  298 IF(V .LE. 0.0) GO TO 300
      XKS(J)=U/V
  300 CONTINUE
```

```
      XKC=XKS(1)
      L=1
C                   SELECT CRITICAL VALUE
      DO 410 N=2,3
      IF(XKC .LE. XKS(N))GO TO 410
      IF(L+N .EQ. 5 .AND. XKC .LE. 1.04*XKS(3)) GO TO 410
      L=N
      XKC=XKS(N)
  410 CONTINUE
      THC=TH(L)
      SC=SA(L)
      RETURN
      END
```

PROGRAM EVAL

```
C
C     PROGRAM TO EVALUATE CHARACTERIZATION MODELS FOR SIMPLE STRESS AND
C     STRAIN HISTORIES
C
      DIMENSION TITLE(12),SEV(6,20),CS(6,6),IFF(6),E(6),XLS(6),
     1 SIG(6),V(6),DVS(6),DE(6),VB(6),DVT(6),DV(6),DVC(6),C(6,6),
     2  XL(6),DSIG(6),DSIGP(6),DEP(6),DEQ(6),DVCP(6),ENVSTF(6),DES(6)
     3  ,DE2(3),CS2(3,3),XLS2(3)
C
C     FORMAT STATEMENT
C
  800 FORMAT (6I1,I4,6E10.2,2F5.1)
  801 FORMAT (12A6)
  802 FORMAT (I1,I4,6E10.2)
  901 FORMAT (1H1, 20X,12A6)
  902 FORMAT (/5X,'STRAIN/STRESS HISTORY CONTROL'/8X,'COMPONENT NO.',
     1    6I10/16X,'IFF =',6I10/13X,'ENVSTF =',6X,1P6E10.2//5X,
     2  'CONVERGENCE FACTOR =',1PE15.3,5X,'CONVERGENCE LIMIT =',1PE15.3)
  903 FORMAT(///5X,'SEGMENT =', I3,3X,'NMIS =',I3,3X,'V1 =',1PE10.2,3X,
     1    'V2 =',1PE10.2,3X,'V3 =',1PE10.2,3X,'V4 =',1PE10.2,3X,'V5 =',
     2    1PE10.2,3X,'V6 =',1PE10.2,///1X,6HINC NO,5X,'SIG-X',6X,'E-X',6X
     3    ,'SIG-Y',6X,'E-Y',6X,'SIG-Z',6X,'E-Z',5X,'SIG-XY',5X,'E-XY',5X,
     4    'SIG-XZ',5X,'E-XZ',5X,'SIG-YZ',5X,'E-YZ'/)
  904 FORMAT(I5,4X,1P12E10.2)
  906 FORMAT (1X,'THE SAMPLE IS NO LONGER STABLE', 4X,'AVG ERROR = ',
     1       E15.5)
  907 FORMAT(1X,'THE SAMPLE HAS COMPLETELY LOST ITS INTEGRITY')
      XLARG = 1.0E+20
   10 ISTLF = 0
      READ (5,801,END=700) TITLE

      WRITE (6,901) TITLE

C              ESTABLISH TYPE OF STRESS AND STRAIN HISTORY
      READ (5,800) (IFF(I),I=1,6),ITYPE,(ENVSTF(I),I=1,6),CONFAC,CONLMT
      WRITE(6,902)(I,I=1,6),(IFF(I),I=1,6),(ENVSTF(I),I=1,6),CONFAC,
     1               CONLMT
      IF(ITYPE .EQ. 0) GO TO 15
C
C     CALL CHARACTERIZATION ROUTINE FOR PURPOSE OF READING IN MATERIAL
C     PROPERTIES
C
      CALL PROP(1,C,0,0,DE,CS,XLS)
      GO TO 20
   15 CALL PROP(1,C,0,0,DE2,CS2,XLS2)
      GO TO 30
C              INITIALIZE ARRAYS
   20 CALL PROP(2,1,0,1,DE,CS,XLS)
      GO TO 35
   30 CALL PROP(2,1,0,1,DE2,CS2,XLS2)
   35 DO 40 I=1,6
      E(I) = 0.0
      SIG(I) = 0.0
      V(I) = 0.0
```

```
      DVS(I) = 0.0
      DE(I) = -0.0001
      IF(I .GT. 3) DE(I) =0.0
   40 CONTINUE
      NMAX = 80
      ICVR = 0
      I = 0
C
C     PROPORTIONATE LOADING SEGMENT LOOP
C
   50 DO 60 J=1,6
   60 VB(J) = V(J)
      KCNT = 0
      READ (5,802) NSEC,NMIS,(V(J),J=1,6)
      IF(NSEC .EQ. 9) GO TO 10
      ISTEP = ISTEP+1
      WRITE(6,903)ISTEP,NMIS,(V(J),J=1,6)
      DU2 = NMIS
      DU1 = 1.0/DU2
      DU = 0.0
      R = 0.0
      DO 70 J=1,6
      DVT(J) = V(J)-VB(J)
      DV(J) = DVT(J)*DU1*1.000001
      DVC(J) = 0.0
C              ESTABLISH STRAIN ESTIMATE FOR FIRST INCREMENT OF
C              LOADING SEGMENT
      IF (DVS(J) .EQ. 0.0) GO TO 70
      R = R+DV(J)/DVS(J)
      DU = DU+1.0
   70 CONTINUE
      IF (DU .EQ. 0.0) GO TO 80
      R = R/DU
      GO TO 90
   80 R = 1.0
   90 IF (R .LT. 0.0) R=0.8*R
      DO 120 K = 1,6
      IF(IFF(K) .NE. 1) GO TO 100
      DE(K) = DV(K)
      GO TO 120
  100 DE(K) = DE(K)*R
  120 CONTINUE
C
C     INCREMENTAL LOADING LOOP
C
  150 DO 180 K=1,6
  180 DES(K)=DE(K)
C
C     ITERATION LOOP
C
      DO 400 J=1,20
      IC = J+ICVR
C              CALL CHARACTERIZATION ROUTINE TO PREDICT INCREMENTAL
C              PROPERTIES
      IF(ITYPE .EQ. 1) GO TO 200
      DE2(1) = DE(1)
      DE2(2) = DE(2)
      DE2(3) = DE(4)
      CALL PROP(3,1,0,IC,DE2,CS2,XLS2)
      DO 185 N=1,6
```

```
      XLS(N) = 0.0
      DO 185 M=1,6
  185 CS(N,M) = 0.0
      DO 190 N=1,2
      XLS(N) = XLS2(N)
      CS(N,4) = CS2(N,3)
      CS(4,N) = CS2(3,N)
      DO 190 M=1,2
  190 CS(N,M) = CS2(N,M)
      XLS(4) = XLS2(3)
      CS(4,4) = CS2(3,3)
      GO TO 230
  200 CALL PROP(3,1,0,IC,DE,CS,XLS)
C               ACCOUNT FOR SPECIFIED STRAINS
  230 DO 250 N=1,6
      DEP(N) = DE(N)
      DO 240 M=1,6
  240 C(N,M) = CS(N,M)
      C(N,N)=C(N,N)+ENVSTF(N)
      XL(N) = XLS(N)
      DSIG(N) = DV(N)-XL(N)
      DSIGP(N) = DV(N)
      IF (IFF(N) .EQ. 0) GO TO 250
      C(N,N) = XLARG
      DSIG(N) = XLARG*DV(N)
  250 DSIGP(N) = DSIG(N)
C               SOLVE FOR INCREMENTAL STRESSES AND STRAINS
      DO 275 N=2,6
      IF(C(N-1,N-1) .NE. 0.0) GO TO 260
      WRITE (6,907)
      GO TO 430
  260 DO 275 JJ=N,6
      R = C(JJ,N-1)/C(N-1,N-1)
      DSIG (JJ) = DSIG (JJ) - R*DSIG (N-1)
      DSIGP(JJ) = DSIGP(JJ) - R*DSIGP(N-1)
      DO 275 K=N,6
  275 C(JJ,K) = C(JJ,K)-R*C(N-1,K)
      DE (6) = DSIG (6)/C(6,6)
      DEQ(6) = DSIGP(6)/C(6,6)
      DO 300 N=2,6
      K = 7-N
      DO 290 JJ=K,5
      DSIG (K) = DSIG (K) - C(K,JJ+1)*DE (JJ+1)
  290 DSIGP(K) = DSIGP(K) - C(K,JJ+1)*DEQ(JJ+1)
      DE (K) = DSIG (K)/C(K,K)
  300 DEQ(K) = DSIGP(K)/C(K,K)
      DU = 0.0
      DO 310 N=1,6
      SEV(N,J) = DEP(N)
  310 DU = DU+ABS(DE(N))
C               CHECK FOR CONVERGENCE
      DUM = 0.0
      DO 320 N=1,6
  320 DUM = DUM+ABS( DE(N)-DEP(N))
      DU = DUM/DU
      JJ = J
      IF(DU .LT. CONLMT) GO TO 440
      DO 330 N=1,6
  330 DE(N)=DEP(N)+CONFAC*(DE(N)-DEP(N))
  400 CONTINUE
```

```
C              FAILURE TO CONVERGE - HALF INCREMENT SIZE
      DO 410 N=1,6
      DV(N) = DV(N)*0.5
  410 DE(N) = DES(N)*0.5
      KONT = KONT+1
      IF (KONT .EQ. 15) GO TO 420
      ICVR = 1
      GO TO 150
C              FAILURE TO CONVERGE EVEN WITH REDUCED INCREMENT SIZE
  420 WRITE (6,906) DU
  430 READ (5,802) NSEC
      IF (NSEC .EQ. 9) GO TO 10
      GO TO 430
C              CONVERGED -ADD INCREMENTAL VALUES TO TOTAL VALUES
C              AND PRINT VALUES
  440 ICVR = 0
      DO 550 N=1,6
      DU = XLS(N)
      DO 500 K=1,6
  500 DU =  DU+CS(N,K)*DE(K)
  550 SIG(N) = SIG(N) + DU
      DO 560 N=1,6
      E(N) = E(N)+DE(N)
      DE(N) = DEO(N)
  560 DVC(N) = DVC(N) + DV(N)
      I = I+1
      WRITE(6,904) I,(SIG(K),E(K),K=1,6)
      IF (I .EQ. NMAX) GO TO 430
      DO 580 N=1,6
  580 DVS(N) = DV(N)
C              CHECK TO SEE IF PROPORTIONAL LOADING SEGMENT IS COMPLETED
      DO 590 N=1,6
      IF(ABS(DVC(N)) .LT. ABS(DVT(N))) GO TO 600
  590 CONTINUE
      GO TO 50
  600 IF(KONT .EQ. 0) GO TO 620
      IF (JJ .GE. 4) GO TO 620
C              IF RAPID CONVERGENCES AND INCREMENT SIZE PREVIOUSLY
C              REDUCED   NOW DOUBLE IT
      DO 610 N=1,6
      DV(N) = DV(N)*2.0
  610 DE(N) = DE(N)*2.0
      KONT=KONT-1
C              MAKE SURE THIS INCREMENT DOES NOT EXCEED SEGMENT SIZE
  620 DO 625 N=1,6
      DVCP(N) = DVC(N)+DV(N)
      IF(ABS(DVCP(N)) .GT. ABS(DVT(N))) DV(N)=(DVT(N)-DVC(N))*1.000001
  625 CONTINUE
      GO TO 150
  700 STOP
      END
```