Supplemental Material

Inference for the Lognormal Mean and Quantiles Based on Samples with Left and Right Type I Censoring

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APPENDIX: DERIVATION OF THE MSLRT STATISTICS

In this Appendix, we provide a detailed derivation of the MSLRT statistics for the lognormal mean and qauntiles, under a type I singly left censored sample.

A1. The MSLRT statistic for the lognormal mean

The mean of a lognormal distribution is given by $\exp(\mu + \frac{\sigma^2}{2})$, where μ and σ^2 are the mean and variance, respectively, of the log-transformed random variable (which follows $N(\mu, \sigma^2)$). Define $\psi = \mu + \frac{\sigma^2}{2}$ and consider the reparametrization $(\mu, \sigma^2) \rightarrow (\psi, \sigma^2)$. Let $(\widehat{\psi}, \widehat{\sigma}^2)$ denote the MLE of (ψ, σ^2), and let $\widehat{\sigma}^2_\psi$ denote the constrained MLE of σ^2 for a fixed ψ. The computation of the MLE is described in Section 2.1, and that of the constrained MLE is explained in Section 3.2.2 in the paper. In particular, $\widehat{\sigma}^2_\psi$ is the solution of equation (13) in the paper. In order to implement the MSLRT, we need to compute the SLRT statistic $R(\psi)$ and the factor $Q(\psi)$ given in equations (12) and (16), respectively, in the paper. Once the MLE $(\widehat{\psi}, \widehat{\sigma}^2)$ and the constrained MLE $\widehat{\sigma}^2_\psi$ are obtained, the computation of $R(\psi)$ is straightforward, using the expression given in (12). Here we shall explain the computation of $Q(\psi)$.

Our derivations are based on the theory described in Wong and Wu (2000); we refer to this article for motivation and further details. In fact the expressions given below are obtained by simplifying the quantities in equations (7)−(17) in Wong and Wu (2000).

Let

$$\theta = (\mu, \sigma^2), \quad \text{and} \quad \psi = \psi(\theta) = \mu + \frac{\sigma^2}{2}.$$

For a function $f(\theta)$, we shall use the notation $f_\theta(\theta)$ to denote the 2×1 vector of first derivatives, and $f_{\theta\theta}(\theta)$ to denote the 2×2 matrix of second derivatives. The likelihood function, as a function of θ, can be expressed as

$$l(\theta) = k \ln \Phi \left(\frac{DL - \mu}{\sigma} \right) - (n-k) \ln \sigma - \frac{1}{2\sigma^2} \sum_{i=1}^{n-k} (X_i - \mu)^2.$$

The MLEs $\widehat{\mu}$ and $\widehat{\sigma}$ are the solutions to the equations

$$\begin{aligned} \frac{\partial l(\theta)}{\partial \mu} &= -\frac{k}{\sigma} \frac{\phi(\xi)}{\Phi(\xi)} + \frac{(n-k)(\bar{X}_l - \mu)}{\sigma^2} = 0 \\ \frac{\partial l(\theta)}{\partial \sigma} &= -\frac{k\xi\phi(\xi)}{\sigma\Phi(\xi)} - \frac{n-k}{\sigma} + \frac{1}{\sigma^3} \sum_{i=1}^{n-k} (X_i - \mu)^2 = 0, \end{aligned} \tag{A.1}$$

where $\xi = \frac{x_0 - \mu}{\sigma}$, as defined in Section 2.1, x_0 being the censoring threshold. Furthermore, the Fisher information matrix $J_\theta(\theta)$ is

$$J_\theta(\theta) = \begin{pmatrix} -\frac{\partial^2 l(\theta)}{\partial \mu^2} & -\frac{\partial^2 l(\theta)}{\partial \mu \partial \sigma} \\ -\frac{\partial^2 l(\theta)}{\partial \mu \partial \sigma} & -\frac{\partial^2 l(\theta)}{\partial \sigma^2} \end{pmatrix}, \tag{A.2}$$

where

$$\begin{aligned} -\frac{\partial^2 l(\mu, \sigma)}{\partial \mu^2} &= \frac{k}{\sigma^2} w(\xi)[w(\xi) + \xi] + \frac{n-k}{\sigma^2} \\ -\frac{\partial^2 l(\mu, \sigma)}{\partial \mu \partial \sigma} &= \frac{k w(\xi)}{\sigma^2} [\xi(\xi + w(\xi)) - 1] + \frac{2(n-k)(\bar{X}_l - \mu)}{\sigma^3} \\ -\frac{\partial^2 l(\mu, \sigma)}{\partial \sigma^2} &= \frac{k\xi w(\xi)}{\sigma^2} [\xi(\xi + w(\xi)) - 2] - \frac{n-k}{\sigma^2} + \frac{3(n-k)}{\sigma^4} (S_l^2 + (\bar{X}_l - \mu)^2), \end{aligned} \tag{A.3}$$

and $w(\xi) = \phi(\xi)/\Phi(\xi)$. Let $\hat{\theta} = (\hat{\mu}, \hat{\sigma})$ denote the MLE, and write

$$V = \begin{pmatrix} 1 & 1 & \dots & 1 \\ \frac{X_1-\widehat{\mu}}{\widehat{\sigma}} & \frac{X_2-\widehat{\mu}}{\widehat{\sigma}} & \dots & \frac{X_{n-k}-\widehat{\mu}}{\widehat{\sigma}} \end{pmatrix} = \begin{pmatrix} V_{11} & V_{12} & \dots & V_{1,n-k} \\ V_{21} & V_{22} & \dots & V_{2,n-k} \end{pmatrix}.$$

Define

$$\begin{aligned} \phi_1(\theta) &= \sum_{j=1}^{n-k} \frac{\partial l(\theta)}{\partial X_j} V_{ij} = -\frac{n-k}{\sigma^2}(\bar{X}_l - \mu) \\ \phi_2(\theta) &= \sum_{j=1}^{n-k} \frac{\partial l(\theta)}{\partial X_j} V_{2j} = -\frac{(n-k)}{\widehat{\sigma}\sigma^2}[S_l^2 + (\bar{X}_l - \widehat{\mu})(\bar{X}_l - \mu)], \end{aligned} \tag{A.4}$$

and

$$\begin{aligned} \phi_\theta(\theta) &= \begin{pmatrix} \frac{\partial \phi_1(\theta)}{\partial \mu} & \frac{\partial \phi_1(\theta)}{\partial \sigma} \\ \frac{\partial \phi_2(\theta)}{\partial \mu} & \frac{\partial \phi_2(\theta)}{\partial \sigma} \end{pmatrix} \\ &= \begin{pmatrix} \frac{n-k}{\sigma^2} & \frac{2(n-k)}{\sigma^3}(\bar{X}_l - \mu) \\ \frac{n-k}{\sigma^2\widehat{\sigma}}(\bar{X}_l - \widehat{\mu}) & \frac{2(n-k)}{\sigma^3\widehat{\sigma}}[S_l^2 + (\bar{X}_l - \mu)(\bar{X}_l - \widehat{\mu})] \end{pmatrix}. \end{aligned} \tag{A.5}$$

For $\phi_\theta(\theta)$ defined above, we note that

$$\begin{aligned} |\phi_\theta(\theta)| &= \frac{2(n-k)^2 S_l^2}{\widehat{\sigma}\sigma^5} \\ \phi_\theta^{-1}(\theta) &= \frac{\sigma^2}{(n-k)S_l^2} \begin{pmatrix} [S_l^2 + (\bar{X}_l - \widehat{\mu})(\bar{X}_l - \mu)] & -\widehat{\sigma}(\bar{X}_l - \mu) \\ -\frac{\sigma(\bar{X}_l - \widehat{\mu})}{2} & \frac{\widehat{\sigma}\sigma}{2} \end{pmatrix}. \end{aligned} \tag{A.6}$$

Let $\psi(\theta) = \mu + \frac{\sigma^2}{2}$ and $\psi_\theta(\theta) = \left(\frac{\partial \psi(\theta)}{\partial \mu}, \frac{\partial \psi(\theta)}{\partial \sigma}\right) = (1, \sigma)$ so that

$$\psi_{\theta\theta}(\theta) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

The constrained MLE of θ, when $\psi(\theta)$ is kept fixed at the value ψ_0, can be obtained by maximizing

$$K(\theta,\alpha) = l(\theta) + \alpha(\psi(\theta) - \psi_0)$$

with respect to α and θ. Let $\widehat{\alpha}$ and $\widehat{\theta}_{\psi_0} = (\widehat{\mu}_{\psi_0}, \widehat{\sigma}_{\psi_0})$ maximize $K(\theta,\alpha)$. We note that $\widehat{\sigma}^2_{\psi_0}$ satisfies equation (13) in the paper, and $\widehat{\mu}_{\psi_0} = \psi_0 - \widehat{\sigma}^2_{\psi_0}/2$. Equating the first derivative of $K(\theta,\alpha)$, with respect to μ, to zero, we also get

$$\widehat{\alpha} = \frac{kw(\widehat{\xi}_0)}{\widehat{\sigma}_{\psi_0}} - (n-k)\frac{(\bar{X}_l - \widehat{\mu}_{\psi_0})}{\widehat{\sigma}^2_{\psi_0}}, \quad \text{with} \quad \widehat{\xi}_0 = \frac{DL - \widehat{\mu}_{\psi_0}}{\widehat{\sigma}_{\psi_0}}.$$

The development of the MSLRT statistic also requires the information matrix, say $\tilde{J}_\theta(\theta)$, based on the "likelihood"

$$\tilde{l}(\theta) = l(\theta) + \widehat{\alpha}(\psi(\theta) - \psi_0).$$

It is clear that

$$\tilde{J}_\theta(\theta) = J_\theta(\theta) - \widehat{\alpha}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

where $J_\theta(\theta)$ is given in (A.2). Define

$$\begin{aligned} \chi(\theta) &= (1, \widehat{\sigma}_{\psi_0})\, \phi_\theta^{-1}(\widehat{\theta}_{\psi_0}) \begin{pmatrix} \phi_1(\theta) \\ \phi_2(\theta) \end{pmatrix} \\ &= \frac{\widehat{\sigma}^2_{\psi_0}}{\sigma^2}\left(\mu - \widehat{\mu}_{\psi_0} - \frac{\widehat{\sigma}^2_{\psi_0}}{2}\right), \end{aligned} \tag{A.7}$$

and

$$\chi(\widehat{\theta}) - \chi(\widehat{\theta}_{\psi_0}) = \frac{\widehat{\sigma}^2_{\psi_0}}{\widehat{\sigma}^2}\left(\widehat{\mu} - \widehat{\mu}_{\psi_0} - \frac{\widehat{\sigma}^2_{\psi_0}}{2}\right) + \frac{\widehat{\sigma}^2_{\psi_0}}{2}. \tag{A.8}$$

To get the above expression, we used the inverse matrix given in (A.6). Define

$$\widehat{\sigma}^2_\chi = (1, \widehat{\sigma}_{\psi_0})\, \tilde{J}_\theta^{-1}(\widehat{\theta}_{\psi_0}) \begin{pmatrix} 1 \\ \widehat{\sigma}_{\psi_0} \end{pmatrix}$$

$$= \frac{J_{\theta,22}(\widehat{\theta}_{\psi_0}) - \widehat{\alpha} - 2J_{\theta,12}(\widehat{\theta}_{\psi_0})\widehat{\sigma}_{\psi_0} + J_{\theta,11}(\widehat{\theta}_{\psi_0})\widehat{\sigma}^2_{\psi_0}}{|\tilde{J}_\theta(\widehat{\theta}_{\psi_0})|} \tag{A.9}$$

where $J_{\theta,ij}(\theta)$ is the (i,j) element of $J_\theta(\theta)$. Then

$$\widehat{\sigma}^2_\chi|\tilde{J}_\theta(\widehat{\theta}_{\psi_0})| = J_{\theta,22}(\widehat{\theta}_{\psi_0}) - \widehat{\alpha} - 2J_{\theta,12}(\widehat{\theta}_{\psi_0})\widehat{\sigma}_{\psi_0} + J_{\theta,11}(\widehat{\theta}_{\psi_0})\widehat{\sigma}^2_{\psi_0}. \tag{A.10}$$

Now let

$$\begin{aligned} Q(\psi_0) &= \operatorname{sign}\left(\widehat{\psi} - \psi_0\right)\left|\chi(\widehat{\theta}) - \chi(\widehat{\theta}_{\psi_0})\right| \left\{ \frac{\left|J_\theta(\widehat{\theta})\right|\left|\phi_\theta(\widehat{\theta}_{\psi_0})\right|^2}{\widehat{\sigma}^2_\chi\left|\tilde{J}_\theta(\widehat{\theta}_{\psi_0})\right|\left|\phi_\theta(\widehat{\theta})\right|^2} \right\}^{\frac{1}{2}} \\ &= \operatorname{sign}(\widehat{\psi} - \psi_0)\left|\frac{\widehat{\sigma}^2_{\psi_0}}{\widehat{\sigma}^2}\left(\widehat{\mu} - \widehat{\mu}_{\psi_0} - \frac{\widehat{\sigma}^2_{\psi_0}}{2}\right) + \frac{\widehat{\sigma}^2_{\psi_0}}{2}\right| \\ &\times \frac{\left|J_\theta(\widehat{\theta})\right|^{\frac{1}{2}}(\widehat{\sigma}/\widehat{\sigma}_{\psi_0})^5}{\left\{J_{\theta,22}(\widehat{\theta}_{\psi_0}) - \widehat{\alpha} - 2J_{\theta,12}(\widehat{\theta}_{\psi_0})\widehat{\sigma}_{\psi_0} + J_{\theta,11}(\widehat{\theta}_{\psi_0})\widehat{\sigma}^2_{\psi_0}\right\}^{1/2}}, \end{aligned} \tag{A.11}$$

where $J_{\theta,ij}(\hat{\theta}_{\psi_0})$ is the (i,j) element of $J_\theta(\hat{\theta}_{\psi_0})$. The quantity $Q(\psi)$ used in Section 3.2.3 of the paper is given by the above expression.

A2. The MSLRT statistic for a lognormal quantile

Here we shall give details of the derivation of the quantity $Q(\eta)$ given in Section 4.2.2. With $\theta = (\mu, \sigma)$, the parameter of interest is $\eta = \eta(\theta) = \mu + z_p\sigma$. Let $\hat{\theta} = (\widehat{\mu}, \widehat{\sigma})$ denote the MLE of θ. Then the MLE of η is $\widehat{\eta} = \widehat{\mu} + z_p\widehat{\sigma}$. Let $\widehat{\sigma}^2_\eta$ denote the constrained MLE of σ^2, obtained by solving equation (21) in the paper. Then the constrained MLE of θ, say $\widehat{\theta}_\eta$, is given by

$$\widehat{\theta}_\eta = (\eta - z_p\widehat{\sigma}_\eta, \widehat{\sigma}_\eta) = (\widehat{\mu}_\eta, \widehat{\sigma}_\eta).$$

The constrained MLE can also be obtained by maximizing $l(\eta, \sigma^2) + \alpha(\eta(\theta) - \eta_0)$, where α is a lagrange multiplier and η_0 is a fixed value of $\eta = \eta(\theta)$. As before, the development of the MSLRT statistic also requires the information matrix, say $\tilde{J}_\theta(\theta)$, based on the "likelihood" $\tilde{l}(\eta, \sigma^2) = l(\eta, \sigma^2) + \widehat{\alpha}(\eta(\theta) - \eta_0)$, $\widehat{\alpha}$ being the value of α that maximizes $l(\eta, \sigma^2) + \alpha(\eta(\theta) - \eta_0)$. Since $\eta(\theta) = \mu + z_p\sigma$, it is easy to verify that $\tilde{J}_\theta(\theta) = J_\theta(\theta)$, the information matrix based on $l(\eta, \sigma^2)$. The latter information matrix is given in equations (A.2) and (A.3).

Define $(\phi_1(\theta), \phi_2(\theta))$ and $\phi_\theta(\theta)$ as in (A.4) and (A.5), respectively, and let

$$\chi(\theta) = (1, z_p)\, \phi_\theta^{-1}(\widehat{\theta}_\eta) \begin{pmatrix} \phi_1(\theta) \\ \phi_2(\theta) \end{pmatrix} = \frac{\widehat{\sigma}_\eta^2}{\sigma^2}\left(\mu - \widehat{\mu}_\eta - \frac{\widehat{\sigma}_\eta z_p}{2}\right). \tag{A.12}$$

It is easy to see that

$$\chi(\widehat{\theta}) - \chi(\widehat{\theta}_\eta) = \frac{\widehat{\sigma}_\eta^2}{\widehat{\sigma}^2}\left(\widehat{\mu} - \widehat{\mu}_\eta - \frac{\widehat{\sigma}_\eta z_p}{2}\right) + \frac{\widehat{\sigma}_\eta z_p}{2}. \tag{A.13}$$

Also define

$$\widehat{\sigma}_\chi^2 = (1, z_p)\, \tilde{J}_\theta^{-1}(\widehat{\theta}_\eta) \begin{pmatrix} 1 \\ z_p \end{pmatrix} = (1, z_p)\, J_\theta^{-1}(\widehat{\theta}_\eta) \begin{pmatrix} 1 \\ z_p \end{pmatrix}, \tag{A.14}$$

where we have used the property $\tilde{J}_\theta(\theta) = J_\theta(\theta)$. With $|\phi_\theta(\theta)|$ as given in equation (A.6), it is easy to see that $|\phi_\theta(\widehat{\theta}_\eta)|/|\phi_\theta(\widehat{\theta}_\eta)| = \widehat{\sigma}^5/\widehat{\sigma}_\eta^5$. Now define

$$\begin{aligned} Q(\eta) &= \operatorname{sign}(\widehat{\eta} - \eta)\left|\chi(\widehat{\theta}) - \chi(\widehat{\theta}_\eta)\right| \times \left\{ \frac{\left|J_\theta(\widehat{\theta})\right|\left|\phi_\theta(\widehat{\theta}_\eta)\right|^2}{\widehat{\sigma}_\chi^2\left|\tilde{J}_\theta(\widehat{\theta}_\eta)\right|\left|\phi_\theta(\widehat{\theta})\right|^2} \right\}^{\frac{1}{2}} \\ &= \operatorname{sign}(\widehat{\eta} - \eta)\left|\frac{\widehat{\sigma}_\eta^2}{\widehat{\sigma}^2}\left(\widehat{\mu} - \widehat{\mu}_\eta - \frac{\widehat{\sigma}_\eta z_p}{2}\right) + \frac{\widehat{\sigma}_\eta z_p}{2}\right| \times \left\{ \frac{1}{\widehat{\sigma}_\chi^2} \times \left(\frac{\widehat{\sigma}^2}{\widehat{\sigma}_\eta^2}\right)^5 \right\}^{\frac{1}{2}}. \end{aligned} \tag{A.15}$$

The quantity $Q(\eta)$ used in Section 4.2.2 of the paper is given by the above expression.

REFERENCE

Wong, A. C. M., and Wu, J. (2000), "Practical Small Sample Asymptotics for Distributions Used in Life Data Analysis," *Technometrics*, 42, 149-155.