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Estimation of Order Restricted Concentration Parameters of von Mises Distributions

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The von Mises distribution is a natural circular analog of the normal distribution on the real line, and is known as the "circular normal distribution". This distribution has two parameters, known as the concentration parameter and the circular mean (or the mean direction). There are practical situations where it is of interest to estimate the concentration parameters of several von Mises distributions, when it is known apriori that the concentration parameters are subject to a simple order restriction. In this article, we discuss the restricted maximum likelihood estimation of the concentration parameters $\kappa_1, \ldots, \kappa_m$ of $m(\geq 2)$ von Mises distributions, when it is known apriori that $0 \leq \kappa_1 \leq \kappa_2 \leq \cdots \leq \kappa_m \leq \infty$. Using the theory of isotonic regression, we derive the restricted maximum likelihood estimators of the concentration parameters. Using approximations of some statistics based on a random sample from the von Mises distributions having large concentration parameters, we propose some more estimators for the order restricted concentration parameters of two von Mises distributions. Using Monte Carlo simulations, the restricted maximum likelihood estimators and the proposed estimators, based on the assumption of large concentration parameters, are compared with the usual (unrestricted) maximum likelihood estimators under the mean squared error criterion.

Keywords Brewster–Zidek technique; Isotonic regression; Maximum likelihood estimator; Mean squared error; von Mises distribution.

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1. Introduction

Problems of statistical inference of order restricted parameters of distributions arise in many practical situations. If it is known apriori that the population parameters satisfy certain order restrictions, then it is natural to incorporate this information in the estimation of the parameters. For a variety of parametric distributions defined on the real line, estimation of the order restricted parameters has been considered in the literature. In the maximum likelihood estimation of ordered normal means with common variance, ordered means of exponential distributions, ordered probabilities of response in binomial distributions, and ordered means of Poisson distributions, the maximum likelihood estimates (MLES) under the order restrictions are the isotonic regression of the usual (unrestricted) MLES with sample sizes as the weights (Robertson et al., 1988). For the case of equal sample sizes, it follows from the theory of isotonic regression that the sum of mean squared errors (MSES) of the isotonic regression estimators is smaller than the sum of MSES of the usual estimators. Lee (1981) was the first to study the component-wise MSE of restricted MLES of ordered normal means and showed that these have a component-wise smaller MSE than the usual MLES (i.e., sample means). Kelly (1989) and Hwang and Peddada (1994) strengthened the results of Lee (1981) by deriving some stochastic dominance results. Kushary and Cohen (1991) established a result similar to Lee's for the maximum likelihood estimation of ordered Poisson means. For the equal sample sizes, Kaur and Singh (1991) established that the restricted MLES of two ordered exponential means have a component-wise smaller MSE compared to the MSE of the sample means. Gupta and Singh (1992) showed that the restricted MLES of two ordered normal means are also more efficient than the sample means with respect to the Pitman nearness criterion. The MLE of the common variance, taking into consideration the order restriction on means, was shown to be better than the usual MLE with respect to the MSE criterion. Sampson et al. (2003) studied the bias issues of the order restricted estimators of two ordered normal means and the estimators of the common variance.

Order restricted estimation procedures have not yet been investigated for circular distributions. Recently, in the problems of modeling torsional angles in molecules (Demchuk and Singh, 2001), it was observed that the von Mises circular distribution fits very well with the angular data of the torsional angle of the methanol molecule. Here it is known a priori that as the temperature increases the concentration of the distribution decreases. This suggests that for the data on the torsional angle of the methanol molecule collected at increasing temperatures, the estimation procedure to estimate the concentration parameters of von Mises distributions should take into consideration the order restriction of the concentration parameters of the corresponding von Mises distributions. This motivates us to consider the problem of maximum likelihood estimation of the order restricted concentration parameters of $m(\geq 2)$ von Mises distributions. In Sec. 2, we derive the MLE of vector of ordered concentration parameters of m von Mises distributions. Based on the approximations of some statistics associated with von Mises distributions having large concentration parameters, seen in Sec. 3, we propose some more estimators for the vector of order restricted concentration parameters of two von Mises distributions. In Sec. 4, using Monte Carlo simulations of equal sample sizes from two von Mises distributions, we compare the MSE of the restricted MLE with those of the unrestricted MLE and the estimators based on the assumption of large concentration parameters. The order restricted

MLE is seen to be superior to the usual MLE with respect to the sum of MSES criterion for the unknown means case. For the known means case, the order restricted MLE has a component-wise smaller MSE compared to the usual MLE for almost all configurations of parameters. The order restricted estimators obtained using properties of some statistics for large concentration parameters also perform reasonably well for moderate to large values of the concentration parameters.

2. Maximum Likelihood Estimation of Ordered Concentration Parameters of m von Mises Distributions

A circular random variable Θ is said to follow a von Mises distribution if its probability density function is given by

$$f(\theta) = \frac{1}{2\pi I_0(\kappa)} e^{\kappa \cos(\theta-\theta_0)}, \quad -\pi < \theta \leq \pi, \ \kappa \geq 0, \ -\pi < \theta_0 \leq \pi, \tag{2.1}$$

where $I_0(\kappa)$ is the modified Bessel function of order 0 and is given by

$$I_0(\kappa) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{\kappa \cos(y)} dy.$$

This distribution was introduced by von Mises (1918). The von Mises distribution has two parameters κ and θ_0. κ is called the concentration parameter and θ_0 is called the mean direction. Note that, for $\kappa = 0$, the von Mises distribution is the same as the uniform distribution on the circle. Since $\cos(\theta - \theta_0) \approx 1 - (\theta - \theta_0)^2/2$ for small values of $|\theta - \theta_0|$, it follows that if the fluctuations around the circular mean θ_0 are small, then the von Mises distribution is approximately normal with mean θ_0 and variance $1/\kappa$, when $\kappa \to \infty$ (refer to Fisher, 1993; Jammalamadaka and SenGupta, 2001; and Mardia and Jupp, 1999, for more details on the von Mises distribution and its applications).

Gumbel et al. (1953) derived the MLEs of κ and θ_0. Let $\Phi_1, \Phi_2, \ldots, \Phi_n$ be a random sample from the von Mises distribution with the probability density function given by (2.1), and let

$$\overline{X} = \frac{1}{n} \sum_{i=1}^{n} \cos(\Phi_i), \quad \overline{Y} = \frac{1}{n} \sum_{i=1}^{n} \sin(\Phi_i).$$

Then the (unrestricted) MLE $\hat{\kappa}$ of κ is given by the solution of the equation

$$A(\hat{\kappa}) = R,$$

where

$$R = \sqrt{\overline{X}^2 + \overline{Y}^2}, \quad A(\kappa) = \frac{I_1(\kappa)}{I_0(\kappa)},$$

and $I_1(\kappa)$ is the modified Bessel function of order 1 and is given by

$$I_1(\kappa) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \cos(\theta) e^{\kappa \cos(\theta)} d\theta.$$

The MLE $\hat{\theta}_0$ of the parameter θ_0 is given by the solution of the equations

$$\overline{X} = R\cos(\hat{\theta}_0), \quad \overline{Y} = R\sin(\hat{\theta}_0).$$

Mardia and Jupp (1999) discussed maximum likelihood estimation of the concentration parameter of the von Mises distribution when the mean of the distribution is known and report an approximation to the bias of the estimator.

In this article, we discuss the maximum likelihood estimation of ordered concentration parameters of $m(\geq 2)$ von Mises distributions. Let Θ_i have the von Mises distribution with probability density function (pdf) given by:

$$f_i(\theta_i) = \frac{1}{2\pi I_0(\kappa_i)} e^{\kappa_i \cos(\theta_i - \theta_{0i})}, \quad -\pi < \theta_i \leq \pi, \ \kappa_i \geq 0, \ -\pi < \theta_{0i} \leq \pi, \tag{2.2}$$

$i = 1, 2, \ldots, m$.

Suppose that it is known apriori that the concentration parameters κ_i satisfy the order restriction $0 \leq \kappa_1, \kappa_2, \leq \cdots \leq \kappa_m < \infty$. Our interest is to estimate the concentration parameters $\kappa_1, \kappa_2, \ldots, \kappa_m$ under this order restriction. First we assume that the circular means θ_{0i}, $i = 1, 2, \ldots, m$, are known and without loss of generality we take them to be zero.

Let $\Theta_{i1}, \Theta_{i2}, \ldots, \Theta_{in_i}$ be a random sample from the distribution of Θ_i, $i = 1, 2, \ldots, m$. Further assume that the m random samples are independent. Let

$$\overline{X}_i = \frac{1}{n_i}\sum_{j=1}^{n_i} \cos(\Theta_{ij}),$$

$i = 1, 2, \ldots, m$. The negative of the natural logarithm of the likelihood function of the observations is given by

$$-\ln L = \left(\sum_{i=1}^{m} n_i\right)\ln(2\pi) + \sum_{i=1}^{m} n_i(\Phi(\kappa_i) - \kappa_i g(i)), \tag{2.3}$$

where

$$\Phi(y) = \ln(I_0(y)) \quad \text{and} \quad g(i) = \overline{X}_i. \tag{2.4}$$

It is well known that $\Phi(y) = \ln I_0(y)$ is a strictly convex function (see Jammalamadaka and SenGupta, 2001, p. 289). For the maximization of (2.3), we need the following lemma from Robertson et al. (1988, p. 48).

Lemma 2.1. *Let $\Phi(\cdot)$ be a real-valued differentiable convex function defined on the real line and let $\phi(\cdot)$ be its derivative. Let $g^*(\cdot)$ be the isotonic regression of a given function $g(\cdot)$ with weight function $w(\cdot)$, with respect to a simple order $\preceq$ defined on $X = \{x_1, x_2, \ldots, x_m\}$, and let $\mathcal{C}$ be the class of isotonic functions on X with respect to the simple order $\preceq$. Then $\phi^{-1}(g^*(\cdot))$ minimizes*

$$\sum_{i=1}^{m}(\Phi(f(x_i)) - f(x_i)g(x_i))w(x_i) \tag{2.5}$$

in the class of functions $f(\cdot) \in C$. The solution is unique if Φ is strictly convex.

The following theorem, which provides the restricted MLEs of the concentration parameters of m von Mises distributions having known circular means and ordered restricted concentration parameters, follows from the above Lemma 2.1, and the discussion on p. 57 and Eq. (1.12) of Barlow et al. (1972).

Theorem 2.1. *Let Θ_{ij}, $j = 1, 2, \ldots, n_i$, be a random sample of size n_i from the distribution with pdf*

$$f_i(\theta_i) = \frac{1}{2\pi I_0(\kappa_i)} e^{\kappa_i \cos(\theta_i)}, \quad -\pi < \theta_i \leq \pi, \ \kappa_i \geq 0, \tag{2.6}$$

$i = 1, 2, \ldots, m$, *and let*

$$\overline{X}_i = \frac{\sum_{j=1}^{n_i} \cos(\Theta_{ij})}{n_i},$$

$i = 1, 2, \ldots, m$. *Suppose that it is known apriori that* $0 \leq \kappa_1 \leq \kappa_2 \leq \cdots \leq \kappa_m < \infty$. *Then, under the order restriction* $0 \leq \kappa_1 \leq \kappa_2 \leq \cdots \leq \kappa_m < \infty$, *the order restricted MLE of* $(\kappa_1, \kappa_2, \ldots, \kappa_m)$ *is* $(\delta_{11}, \delta_{12}, \ldots, \delta_{m1})$, *where*

$$\delta_{i1} = A^{-1}\left(\max_{1 \leq l \leq i} \min_{i \leq h \leq m} \max \left\{ 0, \frac{\sum_{l=l}^{h} n_l \overline{X}_l}{\sum_{l=l}^{h} n_l} \right\} \right), \tag{2.7}$$

$i = 1, 2, \ldots, m$.

Now we consider the situation when the mean directions θ_{0i}, $i = 1, \ldots, m$, are unknown. Let $\Theta_{i1}, \Theta_{i2}, \ldots, \Theta_{in_i}$ be a random sample from the pdf given in (2.2), where $\theta_{01}, \ldots, \theta_{0m}$ are unknown mean directions and $0 \leq \kappa_1 \leq \cdots \leq \kappa_m < \infty$ are order restricted unknown concentration parameters. Further assume that the m samples are independent. Define, for $i = 1, \ldots, m$,

$$\overline{X}_i = \frac{1}{n_i} \sum_{j=1}^{n_i} \cos(\Theta_{ij}), \quad \overline{Y}_i = \frac{1}{n_i} \sum_{j=1}^{n_i} \sin(\Theta_{ij}), \quad \text{and} \quad \overline{R}_i = \sqrt{\overline{X}_i^2 + \overline{Y}_i^2}.$$

Let $\overline{\Theta}_i$, $i = 1, \ldots, m$, be defined by

$$\overline{X}_i = \overline{R}_i \cos(\overline{\Theta}_i), \quad \overline{Y}_i = \overline{R}_i \sin(\overline{\Theta}_i).$$

In this case, the unrestricted MLEs δ^*_{i0} of κ_i, $i = 1, \ldots m$, are given by

$$\delta^*_{i0} = A^{-1}(\overline{R}_i),$$

which is always nonnegative since $A(0) = 0$ and the unrestricted MLE $\hat{\theta}_{0i}$ of θ_{0i} is given by $\overline{\Theta}_i$, $i = 1, \ldots, m$.

Again, the following theorem, which gives order restricted MLEs for the unknown means case, follows from Lemma 2.1.

Theorem 2.2. *Let Θ_{ij}, $j = 1, 2, \ldots, n_i$, be a random sample of size n_i from the distribution with pdf (2.2), $i = 1, \ldots, m$. Suppose that the mean directions are unknown* *and that the unknown concentration parameters satisfy the order restriction* $0 \le \kappa_1 \le \kappa_2 \le \cdots \le k_m < \infty$. *With respect to the usual ordering of real numbers on the set* $X = \{1, 2, \ldots, m\}$ *and with respect to the weight function* $w(i) = n_i$, *defined on* X, *let* $h^*(\cdot)$ *be the isotonic regression of the function* $h(i) = R_i$, $i = 1, 2, \ldots, m$, *defined on* X. *Then, under the order restriction* $0 \le \kappa_1 \le \kappa_2 \le \cdots \le \kappa_m < \infty$, *the order restricted* MLE *of* $(\kappa_1, \kappa_2, \ldots, \kappa_m)$ *is* $(\delta_{11}^*, \delta_{12}^*, \ldots, \delta_{m1}^*)$, *where*

$$\delta_{i1}^* = A^{-1}(h^*(i)) = A^{-1}\left(\max_{1\le j\le i} \min_{i\le h\le m} \frac{\sum_{l=j}^{h} n_l \overline{R}_l}{\sum_{l=j}^{h} n_l} \right), \quad i = 1, 2, \ldots, m.$$

Note that $\delta_{i0}^*, = A^{-1}(\overline{R}_i), i = 1, \ldots, m$, is the unrestricted MLE of κ_i. Using the theory of isotonic regression (see Barlow et al., 1972), it follows that

$$\sum_{i=1}^{m} (\delta_{i2}^* - \kappa_i)^2 n_i < \sum_{i=1}^{m} (\delta_{i0}^* - \kappa_i)^2 n_i, \quad \forall\, 0 \le \kappa_1 \le \cdots \le \kappa_m < \infty,$$

where $\delta_{i2}^* = h_1^*(i)$, is the isotonic regression of $h_1(i) = A^{-1}(\overline{R}_i), i = 1, \ldots, m$, with weights $n_1, n_2, \ldots, n_k$, i.e.,

$$\delta_{i2}^* = \max_{1\le j\le i} \min_{i\le h\le m} \frac{\sum_{i=j}^{h} n_i A^{-1}(\overline{R}_i)}{\sum_{i=j}^{h} n_i}, \tag{2.8}$$

$i = 1, 2, \ldots, m$.

Thus, the total weighted MSE of $(\delta_{12}^*, \ldots, \delta_{m2}^*)$ is smaller than the total weighted MSE of $(\delta_{10}^*, \ldots, \delta_{m0}^*)$. However, it is not known if the total weighted MSE of $(\delta_{11}^*, \ldots, \delta_{m1}^*)$ is smaller than the total weighted MSE of $(\delta_{10}^*, \ldots, \delta_{m0}^*)$.

In the following section, based on approximations for large concentration parameters, we propose some new estimators for the case of two von Mises distributions having order restricted concentration parameters.

3. Estimators Based on Large Concentration Parameters

Throughout this and the following section, we take $m = 2$ and $n_1 = n_2 = n$, say, ($n \ge 6$). We first consider the case when the mean directions θ_{01} and θ_{02} are known and taken to be zero, without loss of generality. In this case, on using approximation (4.8.23) given in Mardia and Jupp (1999), it follows that, for the large values of κ_i, the distribution of $n_i(1 - \overline{X}_i)$ can be approximated by that of a gamma distribution having shape parameter $n_i/2$ and scale parameter $1/\kappa_i$, $i = 1, 2$ (also see Remark 3.1 in Jammalamadaka and SenGupta, 2001, for an intuitive justification of this approximation). Robertson et al. (1988) discussed the maximum likelihood estimation of ordered restricted scale parameters of gamma distributions and, using the above-mentioned asymptotic results for large concentration parameters, it follows from there that the MLEs of κ_1 and κ_2 under the order restriction $0 < \kappa_1 \le \kappa_2$ are given by

$$\delta_{12} = \min\left\{ \frac{1}{2(1 - \overline{X}_1)}, \frac{1}{2 - \overline{X}_1 - \overline{X}_2} \right\}, \tag{3.1}$$

and

$$\delta_{22} = \max\left\{\frac{1}{2(1-\overline{X}_2)}, \frac{1}{2-\overline{X}_1-\overline{X}_2}\right\}, \tag{3.2}$$

respectively. One can derive improved estimators of the inverse of scale parameters using the Brewster–Zidek (1974) technique (see, for example, Vijayashree et al. (1995), for the improved estimation of order restricted scale parameters of gamma distributions). As the derivation of improved estimators of the inverse of scale parameters of gamma distributions, when the scale parameters are order restricted, is similar to the derivation of improved estimators of scale parameters derived by Vijayashree et al. (1995), we will not provide the details of the derivation and will simply provide the improved estimators. For the estimation of k_i based on $\overline{X}_i$ alone, if one uses the result that, for large values of κ_i, $n(1-\overline{X}_i)$ has an approximate gamma distribution with shape parameter $n/2$ and scale parameter $1/\kappa_i$, then it follows that $(2(1-\overline{X}_i))^{-1}$ and $(n-4)(2n(1-\overline{X}_i))^{-1}$ are the MLE and the best scale invariant estimators of κ_i, $i = 1, 2$. Based on the additional information that $0 < \kappa_1 \leq \kappa_2 \leq \infty$ and that the concentration parameter κ_1 is large, if one exploits the Brewster–Zidek technique for the estimation of κ_1, the estimators improving upon $(2(1-\overline{X}_1))^{-1}$ and $(n-4)(2n(1-\overline{X}_1))^{-1}$ are obtained as:

$$\delta_{13} = \min\left\{\frac{n-2}{n(2-\overline{X}_1-\overline{X}_2)}, \frac{1}{2(1-\overline{X}_1)}\right\}, \tag{3.3}$$

and

$$\delta_{14} = \min\left\{\frac{n-2}{n(2-\overline{X}_1-\overline{X}_2)}, \frac{n-4}{2n(1-\overline{X}_1)}\right\}, \tag{3.4}$$

respectively. Similarly, for the estimation of κ_2, the Brewster–Zidek (1974) improvements over the estimators $(2(1-\overline{X}_2))^{-1}$ and $(n-4)(2n(1-\overline{X}_2))^{-1}$ are obtained as:

$$\delta_{23} = \max\left\{\frac{n-2}{n(2-\overline{X}_1-\overline{X}_2)}, \frac{1}{2(1-\overline{X}_2)}\right\}, \tag{3.5}$$

and

$$\delta_{24} = \max\left\{\frac{n-2}{n(2-\overline{X}_1-\overline{X}_2)}, \frac{n-4}{2n(1-\overline{X}_2)}\right\}. \tag{3.6}$$

Now we consider the case when the mean directions are unknown. In this case, using the approximation (4.8.35) given in Mardia and Jupp (1999), it follows that for the large values of κ_i, the distribution of $n_i(1-\overline{R}_i)$ can be approximated by that of the gamma distribution having shape parameter $(n_i-1)/2$ and scale parameter $1/\kappa_i$, $i = 1, 2$. Here, based on these approximations, the MLEs of κ_1 and κ_2 under the order restriction $0 < \kappa_1 \leq \kappa_2$ are given by

$$\delta_{13}^{*} = \min\left\{\frac{n-1}{2n(1-\overline{R}_1)}, \frac{n-1}{n(2-\overline{R}_1-\overline{R}_2)}\right\}. \tag{3.7}$$

and

$$\delta^*_{23} = \min\left\{\frac{n-1}{2n(1-\overline{R}_2)}, \frac{n-1}{n(2-\overline{R}_1-\overline{R}_2)}\right\}, \tag{3.8}$$

respectively. For the estimation of κ_i based on $\overline{R}_i$ alone, if one uses the result that, for large values of κ_i, $n(1-\overline{R}_i)$ has an approximate gamma distribution with shape parameter $(n-1)/2$ and scale parameter $1/\kappa_i$, it follows that $(2(1-\overline{R}_i))^{-1}$ and $(n-5)(2(n-1)(1-\overline{R}_i))^{-1}$ are the MLE and the best scale invariant estimators of κ_i, $i = 1, 2$. Based on the additional information that $0 < \kappa_1 \leq \kappa_2 < \infty$ and that κ_1 is large, if one exploits the Brewster–Zidek technique for the estimation of κ_1,

Table 1
MSES of κ_1, mean known, sample size = 20

$\kappa_1 \backslash \kappa_2$	0.1	0.2	0.5	1	2	5
0.1	0.0920	0.0913	0.0899	0.0907	0.0912	0.0914
	0.0239	0.0288	0.0423	0.0566	0.0603	0.0604
	0.1745	0.1808	0.1940	0.2034	0.2048	0.2048
	0.1452	0.1535	0.1758	0.1982	0.2047	0.2048
	0.1112	0.1130	0.1157	0.1173	0.1173	0.1173
0.2		0.0967	0.0985	0.0972	0.0980	0.0969
		0.0356	0.0499	0.0660	0.0731	0.0722
		0.1205	0.1365	0.1478	0.1521	0.1521
		0.0953	0.1173	0.1410	0.1518	0.1521
		0.0676	0.0720	0.0739	0.0750	0.0750
0.5			0.1181	0.1189	0.1173	0.1169
			0.0701	0.0925	0.1100	0.1105
			0.0306	0.0493	0.0589	0.0595
			0.0176	0.0385	0.0575	0.0595
			0.0110	0.0160	0.0176	0.0178
1				0.1686	0.1664	0.1674
				0.0969	0.1506	0.1673
				0.0446	0.0620	0.0712
				0.0553	0.0553	0.0712
				0.0952	0.0963	0.0981
2					0.4656	0.4668
					0.2071	0.4451
					0.3071	0.4154
					0.3634	0.3960
					0.5388	0.5474
5						3.6208
						1.3656
						1.5525
						1.5873
						2.4017

Table 2
MSES of κ_2, mean known, sample size = 20

$\kappa_1 \backslash \kappa_2$	0.1	0.2	0.5	1	2	5
0.1	0.0924	0.0973	0.1175	0.1698	0.4766	3.6350
	0.0691	0.0735	0.1031	0.1673	0.4766	3.6350
	0.2250	0.1638	0.0599	0.0724	0.4422	3.4715
	0.2100	0.1541	0.0588	0.0730	0.4422	3.4715
	0.1557	0.0994	0.0178	0.0939	0.5528	2.8598
0.2		0.0979	0.1173	0.1695	0.4644	3.6930
		0.0757	0.1006	0.1651	0.4643	3.6930
		0.1685	0.0615	0.0709	0.4324	3.5364
		0.1550	0.0595	0.0718	0.4324	3.5364
		0.1057	0.0191	0.0910	0.5495	2.9207
0.5			0.1165	0.1670	0.4640	3.5967
			0.0977	0.1550	0.4633	3.5967
			0.0695	0.0674	0.4295	3.4412
			0.0616	0.0705	0.4300	3.4412
			0.0264	0.0793	0.5434	2.8620
1				0.1686	0.4695	3.6269
				0.1465	0.4622	3.6269
				0.0604	0.4279	3.4744
				0.0660	0.4339	3.4744
				0.0519	0.5193	2.8912
2					0.4798	3.6377
					0.4596	3.6311
					0.3628	3.4720
					0.3985	3.4790
					0.3307	2.8453
5						3.6487
						3.6524
						3.2797
						3.2930
						1.8748

the estimator improving upon $(2(1-\overline{R}_1))^{-1}$ is obtained as:

$$\delta_{14}^{*} = \min\left\{\frac{n-3}{(n-1)(2-\overline{R}_1-\overline{R}_2)}, \frac{1}{2(1-\overline{R}_1)}\right\}. \tag{3.9}$$

and, for estimation of κ_2, an estimator improving upon $(2(1-\overline{R}_2))^{-1}$ is obtained as:

$$\delta_{24}^{*} = \max\left\{\frac{n-3}{(n-1)(2-\overline{R}_1-\overline{R}_2)}, \frac{1}{2(1-\overline{R}_2)}\right\}. \tag{3.10}$$

Table 3

MSES of estimators of κ_1, mean unknown, sample size = 20

$\kappa_1 \backslash \kappa_2$	0.1	0.2	0.5	1	2	5
0.1	0.1608	0.1596	0.1582	0.1597	0.1593	0.1602
	0.0964	0.0998	0.1214	0.1518	0.1592	0.1602
	0.0986	0.1018	0.1227	0.1521	0.1592	0.1602
	0.2352	0.2370	0.2481	0.2628	0.2664	0.2669
	0.2187	0.2222	0.2454	0.2841	0.3004	0.3014
0.2		0.1274	0.1284	0.1275	0.1293	0.1271
		0.0671	0.0880	0.1171	0.1291	0.1271
		0.0694	0.0897	0.1176	0.1291	0.1271
		0.1567	0.1678	0.1823	0.1881	0.1873
		0.1435	0.1636	0.1986	0.2173	0.2170
0.5			0.1227	0.1238	0.1213	0.1227
			0.0589	0.0964	0.1199	0.1227
			0.0624	0.0981	0.1200	0.1227
			0.0345	0.0526	0.0634	0.0649
			0.0281	0.0544	0.0801	0.0840
1				0.1883	0.1855	0.1872
				0.0911	0.1656	0.1872
				0.0992	0.1676	0.1872
				0.0457	0.0634	0.0741
				0.0459	0.0578	0.0802
2					0.5669	0.5628
					0.2228	0.5323
					0.2663	0.5364
					0.3228	0.4354
					0.3253	0.4244
5						4.5871
						1.6102
						2.0122
						1.6402
						1.5284

For the case when mean directions are unknown, we are not providing the Brewster–Zidek (1974) improvements over the best scale invariant estimators $(n-5)(2(n-1)(1-\overline{R}_i))^{-1}$, $i = 1, 2$, as our simulation study indicated that these improved estimators do not give worth reporting improvements over the estimators in (3.7)–(3.10).

Note that the estimators based on the assumption of large concentration parameters are simple to evaluate as they do not involve computation of $A^{-1}(\cdot)$. The theoretical comparisons of various estimators of k_1 and k_2 seem intractable.

Table 4
MSES of estimators of κ_2, mean unknown, sample size = 20

$\kappa_1 \backslash \kappa_2$	0.1	0.2	0.5	1	2	5
0.1	0.1622	0.1277	0.1218	0.1909	0.5739	4.6210
	0.1949	0.1473	0.1170	0.1870	0.5738	4.6210
	0.1971	0.1489	0.1173	0.1869	0.5738	4.6210
	0.2885	0.2028	0.0668	0.0748	0.4662	3.6895
	0.3055	0.2193	0.0833	0.0828	0.4880	4.2911
0.2		0.1290	0.1219	0.1897	0.5600	4.7897
		0.1536	0.1181	0.1846	0.5599	4.7897
		0.1559	0.1187	0.1846	0.5599	4.7897
		0.2063	0.0679	0.0731	0.4569	3.8472
		0.2208	0.0837	0.0812	0.4767	4.4630
0.5			0.1226	0.1875	0.5553	4.5877
			0.1296	0.1782	0.5545	4.5877
			0.1331	0.1783	0.5545	4.5877
			0.0757	0.0697	0.4497	3.6734
			0.0866	0.0797	0.4698	4.2659
1				0.1881	0.5715	4.6391
				0.1822	0.5645	4.6391
				0.1903	0.5642	4.6391
				0.0624	0.4558	3.7282
				0.0762	0.4841	4.3223
2					0.5790	4.6407
					0.5828	4.6341
					0.6262	4.6337
					0.3840	3.7113
					0.4509	4.3169
5						4.6738
						4.8606
						5.2626
						3.5345
						4.2058

In the following section, using Monte Carlo simulations, we compare the performances of various estimators of κ_1 and κ_2 under the MSE criterion.

4. Monte Carlo Comparisons

For $n_1 = n_2 = n$ (say) and $m = 2$, we use Monte Carlo simulations to compare the MSEs of various estimators discussed in Sec. 2 and 3. For the case of known means (taken to be zero, without loss of generality), the pairs of estimators considered for

Table 5
Sum of MSES of estimators of κ_1 and κ_2, mean known, sample size = 20

$\kappa_1 \backslash \kappa_2$	0.1	0.2	0.5	1	2	5
0.1	0.1844	0.1886	0.2074	0.2605	0.5678	3.7264
	0.0930	0.1023	0.1454	0.2239	0.5369	3.6954
	0.3995	0.3446	0.2539	0.2758	0.6470	3.6763
	0.3552	0.3076	0.2346	0.2712	0.6469	3.6763
	0.2669	0.2124	0.1335	0.2112	0.6701	2.9771
0.2		0.1946	0.2158	0.2667	0.5624	3.7899
		0.1113	0.1505	0.2311	0.5374	3.7652
		0.2890	0.1980	0.2187	0.5845	3.6885
		0.2503	0.1768	0.2128	0.5842	3.6885
		0.1733	0.0911	0.1649	0.6245	2.9957
0.5			0.2346	0.2859	0.5813	3.7136
			0.1678	0.2475	0.5733	3.7072
			0.1001	0.1167	0.4884	3.5007
			0.0792	0.1090	0.4875	3.5007
			0.0374	0.0953	0.5610	2.8798
1				0.3372	0.6359	3.7943
				0.2434	0.6128	3.7942
				0.1050	0.4899	3.5456
				0.1213	0.4892	3.5456
				0.1471	0.6156	2.9893
2					0.9454	4.1045
					0.6667	4.0762
					0.6699	3.8874
					0.7619	3.8750
					0.8695	3.3927
5						7.2695
						5.0180
						4.8322
						4.8803
						4.2765

the simulation study are $(\delta_{1i}, \delta_{2i})$, $i = 0, 1$, defined by

$$\delta_{10} = A^{-1}(\overline{X}_1),$$
$$\delta_{20} = A^{-1}(\overline{X}_2);$$
$$\delta_{11} = A^{-1}\left(\max\left\{0, \min\left\{\overline{X}_1, \frac{\overline{X}_1 + \overline{X}_2}{2}\right\}\right\}\right),$$
$$\delta_{21} = A^{-1}\left(\max\left\{0, \overline{X}_2, \frac{\overline{X}_1 + \overline{X}_2}{2}\right\}\right),$$

and $(\delta_{1i}, \delta_{2i})$, $i = 2, 3, 4$ defined by (3.1)–(3.6).

Table 6
Sum of MSES of estimators of κ_1 and κ_2, mean unknown, sample size = 20

$\kappa_1 \backslash \kappa_2$	0.1	0.2	0.5	1	2	5
0.1	0.3230	0.2873	0.2800	0.3506	0.7332	4.7812
	0.2913	0.2471	0.2384	0.3388	0.7330	4.7812
	0.2957	0.2507	0.2400	0.3390	0.7330	4.7812
	0.5237	0.4398	0.3149	0.3376	0.7326	3.9564
	0.5242	0.4415	0.3287	0.3669	0.7884	4.5925
0.2		0.2564	0.2503	0.3172	0.6893	4.9168
		0.2207	0.2061	0.3017	0.6890	4.9168
		0.2253	0.2084	0.3022	0.6890	4.9168
		0.3630	0.2357	0.2554	0.6450	4.0345
		0.3643	0.2473	0.2798	0.6940	4.6800
0.5			0.2453	0.3113	0.6766	4.7104
			0.1885	0.2746	0.6744	4.7104
			0.1955	0.2764	0.6745	4.7104
			0.1102	0.1223	0.5131	3.7383
			0.1147	0.1341	0.5499	4.3499
1				0.3764	0.7570	4.8263
				0.2733	0.7301	4.8263
				0.2895	0.7318	4.8263
				0.1081	0.5192	3.8023
				0.1221	0.5419	4.4025
2					1.1459	5.2035
					0.8056	5.1664
					0.8925	5.1701
					0.7068	4.1467
					0.7762	4.7413
5						9.2609
						6.4708
						7.2748
						5.1747
						5.7342

For the case unknown means, the pairs of estimators considered for the simulation study are $(\delta^*_{1i}, \delta^*_{2i})$, $i = 0, 1, 2$, defined by

$$\delta^*_{10} = A^{-1}(\overline{R}_1),$$

$$\delta^*_{20} = A^{-1}(\overline{R}_2),$$

$$\delta^*_{11} = A^{-1}\left(\min\left\{\overline{R}_1, \frac{\overline{R}_1 + \overline{R}_2}{2}\right\}\right),$$

$$\delta^*_{21} = A^{-1}\left(\max\left\{\overline{R}_2, \frac{\overline{R}_1 + \overline{R}_2}{2}\right\}\right);$$

Table 7
MSES of estimators of κ_1, mean known, sample size = 50

$\kappa_1 \backslash \kappa_2$	0.1	0.2	0.5	1	2	5
0.1	0.0345	0.0346	0.0348	0.0348	0.0346	0.0350
	0.0117	0.0144	0.0217	0.0250	0.0249	0.0251
	0.1741	0.1795	0.1879	0.1905	0.1905	0.1902
	0.1632	0.1707	0.1849	0.1904	0.1905	0.1902
	0.1479	0.1505	0.1539	0.1549	0.1550	0.1547
0.2		0.0379	0.0375	0.0379	0.0382	0.0378
		0.0198	0.0266	0.0320	0.0325	0.0320
		0.1196	0.1312	0.1354	0.1351	0.1352
		0.1103	0.1271	0.1351	0.1351	0.1352
		0.0972	0.1026	0.1042	0.1040	0.1041
0.5			0.0454	0.0452	0.0462	0.0451
			0.0316	0.0422	0.0459	0.0450
			0.0249	0.0356	0.0377	0.0371
			0.0200	0.0342	0.0377	0.0371
			0.0149	0.0199	0.0208	0.0204
1				0.0589	0.0588	0.0593
				0.0387	0.0582	0.0593
				0.0263	0.0269	0.0275
				0.0303	0.0266	0.0275
				0.0426	0.0402	0.0407
2					0.1417	0.1434
					0.0816	0.1430
					0.2142	0.1987
					0.2384	0.1986
					0.2995	0.2765
5						1.0624
						0.5545
						0.7460
						0.7915
						1.0528

$$\delta^*_{12} = \min\left\{A^{-1}(\overline{R}_1), \frac{A^{-1}(\overline{R}_1) + A^{-1}(\overline{R}_1)}{2}\right\},$$

$$\delta^*_{22} = \max\left\{A^{-1}(\overline{R}_2), \frac{A^{-1}(\overline{R}_1) + A^{-1}(\overline{R}_2)}{2}\right\},$$

and $(\delta^*_{1i}, \delta^*_{2i})$, $i = 3, 4$ defined by (3.7)–(3.10).

The comparisons of MSES are made for different combinations of $\kappa_1, \kappa_2 (0 \leq \kappa_1 \leq \kappa_2)$ and for $n = 20, 50$. For the computation of estimate of MSE of the estimator $\delta_{ij}(\delta^*_{ij})$ of κ_i, we generated observations from appropriate von Mises

Table 8
MSES of estimators of κ_2, mean known, sample size = 50

$\kappa_1 \backslash \kappa_2$	0.1	0.2	0.5	1	2	5
0.1	0.0348	0.0375	0.0451	0.0598	0.1404	1.0739
	0.0267	0.0301	0.0424	0.0598	0.1404	1.0739
	0.2022	0.1420	0.0372	0.0280	0.1982	1.0683
	0.1951	0.1381	0.0370	0.0280	0.1982	1.0683
	0.1750	0.1159	0.0208	0.0412	0.2771	1.1402
0.2		0.0383	0.0448	0.0589	0.1422	1.0749
		0.0306	0.0404	0.0587	0.1422	1.0749
		0.1463	0.0376	0.0276	0.1982	1.0706
		0.1400	0.0370	0.0276	0.1982	1.0706
		0.1218	0.0213	0.0409	0.2762	1.1438
0.5			0.0447	0.0595	0.1409	1.0589
			0.0355	0.0573	0.1409	1.0589
			0.0429	0.0266	0.1971	1.0587
			0.0394	0.0271	0.1971	1.0587
			0.0279	0.0381	0.2755	1.1390
1				0.0588	0.1415	1.0747
				0.0480	0.1411	1.0747
				0.0188	0.1953	1.0743
				0.0217	0.1956	1.0743
				0.0228	0.2721	1.1515
2					0.1417	1.0509
					0.1251	1.0508
					0.1380	1.0480
					0.1568	1.0481
					0.1739	1.1269
5						1.0708
						1.0002
						0.8589
						0.8936
						0.7130

distributions and, for each combination of κ_1, κ_2 and n, $(\delta_{ii} - \kappa_i)^2((\delta^*_{ii} - \kappa_i)^2)$ was computed. The procedure was then repeated 30,000 times to approximate the MSE of $\delta_{ii}(\delta^*_{ii})$ by the average of 30,000 values of $(\delta_{ii} - \kappa_i)^2((\delta^*_{ii} - \kappa_i)^2)$. For different combinations of κ_1, κ_2 $(0 \le \kappa_1 \le \kappa_2)$ and n, these values are tabulated in Tables 1–4 and 7–10. Tables 1 and 7 (Tables 2 and 8) give the MSES of various estimators of $\kappa_1(\kappa_2)$ when the mean directions are known (taken to be zero). Tables 3 and 9 (Tables 4 and 10) give the MSES of various estimators of $\kappa_1(\kappa_2)$ when the mean directions are unknown. In Tables 1 and 7 (Tables 2 and 8), entries from the top to the bottom correspond to the MSES of $\delta_{1i}(\delta_{2i})$, $i = 0, 1, 2, 3, 4$, respectively. In Tables 3 and 9 (Tables 4 and 10), entries from the top to the bottom correspond to the MSES of $\delta^*_{1i}(\delta^*_{2i})$, $i = 0, 1, 2, 3, 4$, respectively.

Table 9
MSES of estimators of κ_1, mean unknown, sample size = 50

$\kappa_1 \backslash \kappa_2$	0.1	0.2	0.5	1	2	5
0.1	0.0499	0.0499	0.0500	0.0497	0.0501	0.0500
	0.0286	0.0322	0.0443	0.0496	0.0501	0.0500
	0.0289	0.0324	0.0444	0.0496	0.0501	0.0500
	0.2082	0.2108	0.2191	0.2223	0.2225	0.2223
	0.2046	0.2089	0.2250	0.2332	0.2336	0.2334
0.2		0.0393	0.0392	0.0394	0.0398	0.0396
		0.0198	0.0317	0.0391	0.0398	0.0396
		0.0201	0.0318	0.0391	0.0398	0.0396
		0.1369	0.1462	0.1508	0.1511	0.1511
		0.1340	0.1498	0.1599	0.1605	0.1606
0.5			0.0451	0.0450	0.0461	0.0452
			0.0263	0.0417	0.0461	0.0452
			0.0267	0.0418	0.0461	0.0452
			0.0263	0.0372	0.0391	0.0387
			0.0248	0.0407	0.0445	0.0441
1				0.0610	0.0612	0.0615
				0.0363	0.0606	0.0615
				0.0372	0.0606	0.0615
				0.0268	0.0273	0.0279
				0.0266	0.0254	0.0262
2					0.1520	0.1536
					0.0811	0.1533
					0.0855	0.1533
					0.2193	0.2031
					0.2197	0.1892
5						1.1729
						0.5683
						0.6137
						0.7611
						0.7349

Tables 5 and 11 give the sum of MSES of pairs of estimators when the mean directions are known (taken to be zero). Entries from the top to bottom correspond to the sum of MSES of the pairs of estimators $(\delta_{1i}, \delta_{2i})$, $i = 0, 1, 2, 3, 4$, respectively. Tables 6 and 12 give the sum of MSES of pairs of estimators when the mean directions are unknown. Entries from the top to bottom correspond to the sum of MSES of the pairs of estimators $(\delta^*_{1i}, \delta^*_{2i})$, $i = 0, 1, 2, 3, 4$, respectively.

The following conclusions are evident from Tables 1–12:

4.1. *Known Means*

(i) For almost all configurations of parameters, the restricted MLE $(\delta_{11}, \delta_{21})$ of (κ_1, κ_2) has a component-wise smaller MSE compared to the unrestricted MLE

Table 10
MSES of estimators of κ_2, mean unknown, sample size = 50

$\kappa_1 \backslash \kappa_2$	0.1	0.2	0.5	1	2	5
0.1	0.0502	0.0401	0.0449	0.0619	0.1502	1.1864
	0.0605	0.0429	0.0422	0.0618	0.1502	1.1864
	0.0608	0.0430	0.0422	0.0618	0.1502	1.1864
	0.2337	0.1587	0.0389	0.0283	0.2025	1.0901
	0.2374	0.1633	0.0439	0.0267	0.1885	1.1236
0.2		0.0398	0.0448	0.0611	0.1520	1.1906
		0.0456	0.0414	0.0609	0.1520	1.1906
		0.0460	0.0414	0.0609	0.1520	1.1906
		0.1616	0.0393	0.0280	0.2025	1.0954
		0.1646	0.0441	0.0263	0.1887	1.1288
0.5			0.0445	0.0619	0.1508	1.1684
			0.0416	0.0598	0.1508	1.1684
			0.0420	0.0597	0.1508	1.1684
			0.0446	0.0269	0.2016	1.0807
			0.0467	0.0259	0.1877	1.1117
1				0.0612	0.1520	1.1928
				0.0540	0.1515	1.1928
				0.0550	0.1515	1.1928
				0.0191	0.1999	1.1021
				0.0210	0.1866	1.1346
2					0.1514	1.1626
					0.1402	1.1625
					0.1446	1.1625
					0.1413	1.0716
					0.1489	1.1031
5						1.1814
						1.1491
						1.1945
						0.8771
						0.9682

$(\delta_{10}, \delta_{20})$. Thus, for the component-wise estimation of κ_1 and/or κ_2 under the prior information $0 \le \kappa_1 \le \kappa_2 < \infty$, the restricted MLE should be preferred over the unrestricted MLE.

(ii) For all configurations of parameters, the sum of the MSES of restricted MLES δ_{11} and δ_{21} is smaller than the sum of the MSES of unrestricted MLES of δ_{10} and δ_{20}. Thus, in practical situations, if one has the prior information of ordering between the concentration parameters, then the restricted MLE $(\delta_{11}, \delta_{21})$ should be preferred over unrestricted MLE $(\delta_{10}, \delta_{20})$ for the simultaneous estimation of (κ_1, κ_2).

(iii) There is no pair of order restricted estimators which is best with respect to the criterion of component-wise MSE or with respect to the criterion of sum of MSES.

Table 11
Sum of MSES of estimators of κ_1 and κ_2, mean known, sample size = 50

$\kappa_1 \backslash \kappa_2$	0.1	0.2	0.5	1	2	5
0.1	0.0693	0.0721	0.0799	0.0946	0.1750	1.1089
	0.0384	0.0445	0.0641	0.0848	0.1653	1.0990
	0.3763	0.3215	0.2251	0.2185	0.3887	1.2585
	0.3583	0.3088	0.2219	0.2184	0.3887	1.2585
	0.3229	0.2664	0.1747	0.1961	0.4321	1.2949
0.2		0.0762	0.0823	0.0968	0.1804	1.1127
		0.0504	0.0670	0.0907	0.1747	1.1069
		0.2659	0.1688	0.1630	0.3333	1.2058
		0.2503	0.1641	0.1627	0.3333	1.2058
		0.2190	0.1239	0.1451	0.3802	1.2479
0.5			0.0901	0.1047	0.1871	1.1040
			0.0671	0.0995	0.1868	1.1039
			0.0678	0.0622	0.2348	1.0958
			0.0594	0.0613	0.2348	1.0958
			0.0428	0.0580	0.2963	1.1594
1				0.1177	0.2003	1.1340
				0.0867	0.1993	1.1340
				0.0451	0.2222	1.1018
				0.0520	0.2222	1.1018
				0.0654	0.3123	1.1922
2					0.2834	1.1943
					0.2067	1.1938
					0.3522	1.2467
					0.3952	1.2467
					0.4734	1.4034
5						2.1332
						1.5547
						1.6049
						1.6851
						1.7658

(iv) The estimators based on the assumption of large concentration parameters seem to perform reasonably well for moderate to large values of κ_1 and κ_2. The Brewster–Zidek method has helped in improving the performance. Since these estimators are also simple to use, they could be used in practical situations where one has additional prior information that the concentration parameter κ_1 is large.

4.2. *Unknown Means*

(i) For all configurations of parameters, the restricted MLE $(\delta^*_{11}, \delta^*_{21})$ is superior to the unrestricted MLE $(\delta^*_{10}, \delta^*_{20})$ with respect to the sum of the MSES criterion.

Table 12
Sum of MSES of estimators of κ_1 and κ_2, mean unknown, sample size = 50

$\kappa_1 \backslash \kappa_2$	0.1	0.2	0.5	1	2	5
0.1	0.1001	0.0900	0.0949	0.1116	0.2003	1.2364
	0.0891	0.0751	0.0865	0.1114	0.2003	1.2364
	0.0897	0.0754	0.0866	0.1114	0.2003	1.2364
	0.4419	0.3695	0.2580	0.2506	0.4250	1.3124
	0.4420	0.3722	0.2689	0.2599	0.4221	1.3570
0.2		0.0791	0.0840	0.1005	0.1918	1.2302
		0.0654	0.0731	0.1000	0.1918	1.2302
		0.0661	0.0732	0.1000	0.1918	1.2302
		0.2985	0.1855	0.1788	0.3536	1.2465
		0.2986	0.1939	0.1862	0.3492	1.2894
0.5			0.0896	0.1069	0.1969	1.2136
			0.0679	0.1015	0.1969	1.2136
			0.0687	0.1015	0.1969	1.2136
			0.0709	0.0641	0.2407	1.1194
			0.0715	0.0666	0.2322	1.1558
1				0.1222	0.2132	1.2543
				0.0903	0.2121	1.2543
				0.0922	0.2121	1.2543
				0.0459	0.2272	1.1300
				0.0476	0.2120	1.1608
2					0.3034	1.3162
					0.2213	1.3158
					0.2301	1.3158
					0.3606	1.2747
					0.3686	1.2923
5						2.3543
						1.7174
						1.8082
						1.6382
						1.7031

Thus, in practical situations, if one has the prior information of ordering between the concentration parameters, then the restricted MLE $(\delta_{11}^*, \delta_{21}^*)$ should be preferred over unrestricted MLE $(\delta_{10}^*, \delta_{20}^*)$ for the simultaneous estimation of (κ_1, κ_2).

(ii) As expected, for all configurations of parameters, the estimator $(\delta_{12}^*, \delta_{22}^*)$ is superior to the unrestricted MLE $(\delta_{10}^*, \delta_{20}^*)$ with respect to the criterion of sum of MSES. Also, for all configurations of parameters, the restricted MLE $(\delta_{11}^*, \delta_{21}^*)$ is superior to $(\delta_{12}^*, \delta_{22}^*)$ with respect to the criterion of sum of MSES.

(iii) Again there is no pair of order restricted estimators which is uniformly best with respect to the criterion of component-wise MSE or with respect to the criterion of sum of MSES.

(iv) The estimators based on the assumption of large concentration parameters seem to perform reasonably well for moderate to large values of κ_1 and κ_2. Since these estimators are also simple to use, they could be used in practical situations where one has additional prior information that the concentration parameter κ_1 is large.

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