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Stochastic comparisons of Poisson and binomial random variables with their mixtures

Neeraj Misra[a,b], Harshinder Singh[a,c,*], E. James Harner[a]

[a]Department of Statistics, West Virginia University, PO Box 6330, Morgantown, WV 26506-6330, USA
[b]Department of Mathematics, Indian Institute of Technology Kanpur, Kanpur 208 016, India
[c]Health Effects Laboratory, National Institute for Occupational Safety and Health, Morgantown, WV 26505-2888, USA

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Abstract

Motivated by an ecological sampling problem, we compare a Poisson distribution having a fixed mean with a Poisson distribution having a random mean, which has an arbitrary continuous (or discrete) probability distribution. These comparisons are made with respect to the likelihood ratio ordering, simple stochastic ordering, uniform variability ordering and expectation ordering. As a particular case, the mixed Poisson and the Poisson distribution with a fixed mean are compared when both the distributions have the same mean. Similar comparisons are made between the mixed binomial and the binomial distribution having a fixed probability of success.

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1. Introduction

This paper is motivated by an ecological sampling problem. Specifically, the abundance of intertidal rock-surface species is determined by counting the number of each species present in sampled quadrants of a specified size (e.g., $0.5\ m^2$). Since it is expensive and time-consuming to identify and count the individuals in each of a large number of species that can be present in a study, sub-sampling is often done. Assuming the abundance of a species is Poisson distributed, the issue

* Corresponding author. Department of Statistics, West Virginia University, PO Box 6330, Morgantown, WV 26506-6330, USA. Fax: 304-293-2272.

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arises as to whether it is better to mix the sample and divide out $\frac{1}{4}$ (say) of the volume to be counted (a fixed Poisson) or to divide the quadrant into fourths and randomly sample one sub-quadrant (a mixed Poisson). We want to study the effects of the two sub-sampling approaches by comparing a species count N (fixed Poisson) to M (mixed Poisson) in terms of various stochastic orders. Since too many 0 counts can cause estimation problems (e.g., in a generalized linear model), as a special case, we want to study the effects of the two sub-sampling approaches by comparing the probability of getting at least one count of a species under the two approaches.

Furthermore, in categorical data analysis, where the Poisson or binomial sampling is assumed, the data sometimes displays more (or less) variability than is predicted for these models. It might happen because the true sampling distribution is a mixture of different Poisson or binomial distributions (Agresti, 1990, p. 42). Thus, in real-life situations, it is of interest to compare Poisson and binomial distributions with their mixtures with respect to the variability and other stochastic orderings.

In situations such as described above, it is known that the probability density function (pdf) of some variable X of interest is a mixture of pdfs of random variables Y_θ, $\theta \in \Omega$, and it is of interest of compare random variables X and Y_μ, for a fixed $\mu \in \Omega$. In particular, it may be of interest to compare random variables X and $Y_{\bar{\theta}}$, where $\bar{\theta}$ is the mean of the mixing distribution. Shaked (1980) made such comparisons when the pdfs of random variables Y_θ, $\theta \in \Omega$, belong to the general exponential family. He showed that the two resulting pdfs must cross each other exactly twice in a prescribed manner and, as a consequence of this finding, he established the variability ordering between the two random variables. In this paper, we consider mixtures of two specific members (namely, the Poisson and binomial) of the exponential family and compare them with any member in that family with respect to various stochastic orderings.

In Section 2, we define various stochastic orders used in this paper for making comparisons of Poisson and binomial random variables to their mixtures. With respect to these stochastic orders, in Section 3, we derive conditions which ensure that the mixed Poisson random variable, with an arbitrary mixing distribution, is larger than the Poisson random variable having a fixed mean. Similar comparisons between the mixed binomial random variable, with an arbitrary mixing distribution, and the binomial random variable having a fixed mean are made in Section 4.

2. Orderings of distributions

Stochastic orders are useful in comparing random variables measuring certain characteristics in diverse areas. The simplest comparison is through comparing the means of the two random variables (expectation ordering). However, this comparison is not very informative, being based only on the two numbers. A more informative comparison is through the various stochastic orders introduced in the probability literature. Below, we define some of the stochastic orders which are relevant in the context of this paper. For more details, we refer to Ross (1983), Singh (1989) and Shaked and Shanthikumar (1994).

Likelihood ratio ordering: Let X and Y be two random variables having probability density functions (pdfs) $f_X(.)$ and $f_Y(.)$, respectively. The random variable X is said to be larger than the random variable Y in the likelihood ratio ordering (written as $X \geqslant_{\rm lr} Y$) if, for all real numbers x and y ($y \leqslant x$), $f_X(x)f_Y(y) \geqslant f_X(y)f_Y(x)$. This is the strongest ordering between the two random variables.

Simple stochastic order: Let $F_X(x)$ and $F_Y(x)$ be the distribution functions of random variables X and Y, respectively. The random variable X is said to be larger than the random variable Y in the simple stochastic order (written as $X \geqslant_{\rm st} Y$) if $F_X(x) \leqslant F_Y(x)$, for all real numbers x. This is the most commonly used stochastic order in the area of probability and statistics and is usually called the stochastic order.

If $X \geqslant_{\rm st} Y$, then $E(X) \geqslant E(Y)$, where $E(.)$ denotes the expectation. Also, $X \geqslant_{\rm lr} Y$ implies that $X \geqslant_{\rm st} Y$ (Ross, 1983) but, in general, the converse may not be true. If X is larger than Y in any of the above two orderings, it implies that X is likely to take higher values compared to Y.

Expectation ordering: The random variable X is said to be larger than the random variable Y in the expectation ordering (written as $X \geqslant_{\rm E} Y$) if $E(X) \geqslant E(Y)$. This is the simple ordering between means of the two distributions.

Convex ordering: The random variable X is said to be larger than the random variable Y in the convex ordering (written as $X \geqslant_{\rm cx} Y$) if, for every real valued convex function $\phi(.)$ defined on the real line, $E(\phi(X)) \geqslant E(\phi(Y))$.

If the random variable X is larger than the random variable Y in the convex ordering, then $E(X) = E(Y)$ and X is likely to be more variable than Y. This ordering implies that $\mathrm{Var}(X)$ is larger than $\mathrm{Var}(Y)$ (Shaked and Shanthikumar, 1994).

Uniformly more variable ordering: Let X and Y be two random variables with pdfs $f_X(.)$ and $f_Y(.)$, respectively, and let $\mathrm{supp}(X)$ and $\mathrm{supp}(Y)$ denote the respective supports. X is said to be uniformly more variable than Y (denoted as $X \geqslant_{\rm uv} Y$) if $\mathrm{supp}(Y) \subseteq \mathrm{supp}(X)$ and the ratio $f_Y(x)/f_X(x)$ is unimodal over $\mathrm{supp}(X)$ but X and Y are not ordered by the usual stochastic order.

For random variables X and Y having the same mean, it is known that $X \geqslant_{\rm uv} Y$ implies that $X \geqslant_{\rm cx} Y$ (Shaked and Shanthikumar, 1994).

3. Stochastic orderings between Poisson and mixed Poisson distributions

Let N be a random variable having Poisson distribution with a fixed mean $\lambda > 0$, i.e.,

$$\Pr(N = k) = \frac{e^{-\lambda}\lambda^k}{k!}, \quad k = 0, 1, 2, \ldots . \tag{1}$$

Let M be a Poisson random variable with mean Θ, where Θ is a non-degenerate random variable having the probability density function $g(\theta)$, $\theta > 0$. The assumption that Θ is non-degenerate is not restrictive as, in the degenerate case, N and M both will have the Poisson distribution and various orderings between them follow the ordering between their means. Also, although we have assumed Θ to be a continuous random variable, all the results of this section hold even when Θ is assumed to be an arbitrary discrete random variable. We have

$$\Pr(M = k) = \int_0^\infty \frac{e^{-\theta}\theta^k}{k!} g(\theta)\, d\theta, \quad k = 0, 1, \ldots . \tag{2}$$

In this section, we make comparisons between the random variables N and M with respect to the likelihood ratio, stochastic, uniform variability and expectation orderings. The following lemma will be useful in proving the main results.

Lemma 3.1. *Define*, $a_1(k) = E(e^{-\Theta}\Theta^{k+1})/E(e^{-\Theta}\Theta^k)$, $k = 0, 1, \ldots$, $\lambda_0 = a_1(0)$, $\lambda_1 = -\ln[E(e^{-\Theta})]$, $\lambda_2 = E(\Theta)$ *and* $h_k(x) = x\sum_{j=0}^{k}[(-1)^j(\ln x)^j/j!]$, $0 < x < 1$, $k = 0, 1, \ldots$. *Then*,

(a) $a_1(k)$ *is an increasing function of* $k \in \{0, 1, \ldots\}$,
(b) *for each fixed* $k \in \{0, 1, \ldots\}$, $h_k(x)$ *is a concave function of* $x \in (0, 1)$,
(c) $0 < \lambda_0 \leqslant \lambda_1 \leqslant \lambda_2$.

Proof. (a) We may write

$$a_1(k) = \frac{\int_0^\infty e^{-\theta}\theta^{k+1}g(\theta)\,d\theta}{\int_0^\infty e^{-\theta}\theta^{k}g(\theta)\,d\theta} = E(W_k), \quad k = 0, 1, \ldots,$$

where W_k is a random variable having the pdf

$$\psi_k(x) = C_k e^{-x}x^k g(x), \quad x > 0;$$

here C_k is the normalizing constant.

Fix $k \in \{0, 1, \ldots\}$. Clearly, $\psi_{k+1}(x)/\psi_k(x)$ is an increasing function of $x > 0$, which implies that $W_{k+1} \geqslant_{\rm lr} W_k$, which further implies that $W_{k+1} \geqslant_{\rm st} W_k$ and therefore $a_1(k+1) = E(W_{k+1}) \geqslant E(W_k) = a_1(k)$. Since the chosen $k \in \{0, 1, \ldots\}$ was arbitrary, the assertion follows.

(b) For $k = 0$, note that $h_0(x) = x$ is both concave and convex. For $k \in \{1, 2, \ldots\}$, we have

$$\frac{d^2}{dx^2}h_k(x) = -\frac{(-\ln x)^{k-1}}{x(k-1)!} < 0, \quad \forall\ 0 < x < 1$$

which proves the assertion (b).

(c) $\lambda_0 \leqslant \lambda_1$ follows using the concavity of the function $d_1(x) = -x\ln x$, $0 < x < 1$, and Jensen's inequality. Similarly, $\lambda_1 \leqslant \lambda_2$ follows using the concavity of the function $d_2(x) = \ln x$, $0 < x < 1$, and Jensen's inequality. □

The following theorem provides conditions on parameters λ, λ_0, λ_1 and λ_2 that ensure that M is larger than N in various orderings.

Theorem 3.1. *Let* N *and* M *be random variables having distributions given by* (1) *and* (2), *respectively. Then, under the notations of Lemma* 3.1,

(a) $M \geqslant_{\rm lr} N$ *if and only if* $0 < \lambda \leqslant \lambda_0$,
(b) $M \geqslant_{\rm st} N$ *if and only if* $0 < \lambda \leqslant \lambda_1$,
(c) $M \geqslant_{\rm E} (\leqslant_{\rm E}) N$ *if and only if* $0 < \lambda \leqslant (\geqslant) \lambda_2$.

Proof. (a) Consider the likelihood ratio

$$l(k) = \frac{\Pr(M = k)}{\Pr(N = k)} = \frac{E(e^{-\Theta}\Theta^k)}{e^{-\lambda}\lambda^k}, \quad k = 0, 1, 2, \ldots . \tag{3}$$

Then

$$l(k+1) \geqslant l(k)\ \forall\ k = 0, 1, \ldots,$$

$$\Leftrightarrow \lambda \leqslant \frac{E(\mathrm{e}^{-\Theta}\Theta^{k+1})}{E(\mathrm{e}^{-\Theta}\Theta^{k})}, \quad \forall\ k = 0, 1, \ldots$$

$$\Leftrightarrow \lambda \leqslant \lambda_0, \quad \text{(by virtue of Lemma 3.1 (a))},$$

which provides a necessary and sufficient condition for the ratio in (3) to be non-decreasing. Hence the assertion (a) follows.

(b) First, suppose that $0 < \lambda \leqslant \lambda_1$. For, $k = 0, 1, 2, \ldots$, consider

$$\begin{aligned}\Delta_\lambda(k) &= \Pr(N \leqslant k) - \Pr(M \leqslant k)\\ &= \mathrm{e}^{-\lambda}\sum_{j=0}^{k}\frac{\lambda^j}{j!} - \sum_{j=0}^{k}\frac{E(\mathrm{e}^{-\Theta}\Theta^j)}{j!}.\end{aligned}$$

Since the Poisson distribution is stochastically increasing in the mean λ, to show that $M \geqslant_{\mathrm{st}} N$, it is enough to establish that

$$\Delta_{\lambda_1}(k) \geqslant 0, \quad \forall\ k = 0, 1, 2, \ldots .$$

For fixed $k \in \{0, 1, 2, \ldots\}$, then

$$\Delta_{\lambda_1}(k) = E(\mathrm{e}^{-\Theta})\sum_{j=0}^{k}(-1)^j\frac{[\ln\{E(\mathrm{e}^{-\Theta})\}]^j}{j!} - \sum_{j=0}^{k}\frac{E(\mathrm{e}^{-\Theta}\Theta^j)}{j!}.$$

By applying Jensen's inequality to the function $h_k(x)$, defined in Lemma 3.1, it follows from above that

$$\Delta_{\lambda_1}(k) \geqslant 0.$$

Conversely, suppose that $M \geqslant_{\mathrm{st}} N$. Then

$$\begin{aligned}\Pr(N \leqslant 0) &\geqslant \Pr(M \leqslant 0)\\ &\Leftrightarrow \lambda \leqslant \lambda_1.\end{aligned}$$

Hence the assertion (b) is proved.

(c) This is obvious. □

Remarks. 1. We note that $\Pr(M \geqslant 1) \geqslant \Pr(N \geqslant 1) \Leftrightarrow \Pr(M=0) \leqslant \Pr(N=0) \Leftrightarrow \lambda \leqslant \lambda_1 \Leftrightarrow M \geqslant_{\mathrm{st}} N$, as proved in the above theorem. Thus, it is interesting to observe that $\Pr(M \geqslant 1) \geqslant \Pr(N \geqslant 1)$ ensures that $M \geqslant_{\mathrm{st}} N$.

2. From Lemma 3.1 (c) we have $\lambda_0 \leqslant \lambda_1 \leqslant \lambda_2$ and it follows from the above theorem that M is larger than N in the likelihood ratio ordering when λ falls in the interval $(0, \lambda_0]$. For $\lambda \in (\lambda_0, \lambda_1]$, the random variable M is larger than N in stochastic ordering but, in this range of λ, M is not larger then N in the likelihood ratio ordering. For $\lambda \in (\lambda_1, \lambda_2]$, the random variable M is larger than the random variable N in the expectation ordering but, in this range of λ, M is not larger than N in the simple stochastic ordering.

3. Theorem 3.1 (b) provides a condition under which sampling from the mixture of Poisson distributions is more favorable than the sampling from Poisson distribution.

4. When the mixing distribution belongs to the gamma family, it can be shown that the distribution of the mixture random variable M is negative binomial, which is a common species abundance distribution. Interestingly, in this case, it can be shown that $\lambda \leqslant \lambda_0 \Leftrightarrow \mathrm{CV}(N) \geqslant \mathrm{CV}(M)$, where $\mathrm{CV}(.)$ denotes the coefficient of variation. Thus, in this case, $M \geqslant_{\mathrm{lr}} N \Leftrightarrow \mathrm{CV}(N) \geqslant \mathrm{CV}(M)$.

In the following theorem, we establish that if the mixing random variable Θ is such that, for every $x > 0$, $\Pr(\Theta > x) > 0$, then no value of $\lambda > 0$ can ensure $N \geqslant_{\mathrm{st}} M$. Under this condition on the mixing random variable Θ, in the following theorem, we also establish that $M \geqslant_{\mathrm{uv}} N$ if and only if $\lambda > \lambda_1$.

Theorem 3.2. *Suppose that, for every $x > 0$, $\Pr(\Theta > x) > 0$, then*

(a) *no value of $\lambda > 0$ can ensure $N \geqslant_{\mathrm{st}} M$,*
(b) *$M \geqslant_{\mathrm{uv}} N$ if and only if $\lambda > \lambda_1$.*

Proof. (a) Using the relationship between the Poisson and gamma probabilities, we may write

$$\Pr(N \geqslant k) = \int_0^\lambda \frac{e^{-y} y^{k-1}}{(k-1)!}\,dy, \quad k = 1, 2, \ldots \tag{4}$$

and

$$\Pr(M \geqslant k) = \int_0^\infty \bar{G}(y) \frac{e^{-y} y^{k-1}}{(k-1)!}\,dy, \quad k = 1, 2, \ldots, \tag{5}$$

where $\bar{G}(y) = \int_y^\infty g(t)\,dt$, $y > 0$, denotes the survival function of the mixing random variable Θ. From (4), it follows that

$$\Pr(N \geqslant k) \leqslant \frac{\lambda^k}{k!}, \quad k = 1, 2, \ldots.$$

Choose $\mu > \lambda$. Then, from (5), it follows that

$$\begin{aligned}\Pr(M \geqslant k) &\geqslant \int_0^\mu \bar{G}(y) \frac{e^{-y} y^{k-1}}{(k-1)!}\,dy \\ &\geqslant e^{-\mu}\bar{G}(\mu)\frac{\mu^k}{k!}, \quad k = 1, 2, \ldots.\end{aligned}$$

Therefore,

$$\begin{aligned}R(k) &= \frac{\Pr(N \geqslant k)}{\Pr(M \geqslant k)} \\ &\leqslant \frac{(\lambda/\mu)^k}{\bar{G}(\mu)e^{-\mu}} \\ &\to 0, \quad \text{as } k \to \infty.\end{aligned}$$

Thus it follows that, for any $\lambda > 0$, there exists a sufficiently large k (depending on λ) such that $\Pr(N \geqslant k) < \Pr(M \geqslant k)$, which proves the assertion (a).

(b) Suppose that $\lambda > \lambda_1$. Then, from Theorem 3.1 (b) and (a) above, it is clear that random variables M and N are not ordered by the simple stochastic order. Also, it follows from the arguments used in the proof of Theorem 3.1 (a) that $\Pr(N = k)/\Pr(M = k)$ is unimodal, implying that $M \geqslant_{\text{uv}} N$. The converse part follows by using the similar arguments. □

In the following theorem, we compare the random variables N and M, when $E(\Theta) = \lambda$. In this case, N and M have the same mean i.e., $\lambda = E(N) = E(\Theta) = E(M) = \lambda_2$.

Theorem 3.3. *Suppose that* $\lambda_2 = E(\Theta) = \lambda$. *Then,*

(a) $M \geqslant_{\text{uv}} N$,
(b) $M \geqslant_{\text{cx}} N$, *which further implies that* $\text{Var}(N) \leqslant \text{Var}(M)$,
(c) $\Pr(N \geqslant 1) > \Pr(M \geqslant 1)$,
(d) *under the assumption of Theorem* 3.2, *neither* M *is larger than* N *in the stochastic ordering nor* N *is larger than* M *in the stochastic ordering.*

Proof. (a) Follows from Theorem 3.2 (b), since $\lambda_2 > \lambda_1$.

(b) Since $M \geqslant_{\text{uv}} N$ and $E(M) = E(N)$, it implies that $M \geqslant_{\text{cx}} N$. Thus $E((\phi(N)) \leqslant E(\phi(M))$ for all convex functions $\phi(.)$. Thus $E(M^2) \geqslant E(N^2)$, which implies that $\text{Var}(M) \geqslant \text{Var}(N)$, since M and N have the same means.

(c) It is easily seen that $\Pr(N = 0) \leqslant \Pr(M = 0)$, if $\lambda \geqslant \lambda_1$. The proof follows by observing that $\lambda_2 \geqslant \lambda_1$.

(d) From (c), it follows that M cannot be larger than N in the simple stochastic order.

The fact that N cannot be larger than M in the simple stochastic order is established in Theorem 3.2 (a). □

Remarks. 1. Results (a) and (b) of Theorem 3.3 are proved under a more general setting by Shaked (1980).

2. For the sampling of rare species in ecology, Theorem 3.3 (c) suggests that if the Poisson distribution has the same mean as that of the mixing distribution, then the sampling from the Poisson distribution is more favorable than the sampling from mixture of Poisson distributions. A similar criterion based on the probability of finding at least one fault is used in software engineering to compare partition and random testing procedures (Boland et al., 2002; Weyuker and Jeng, 1991).

4. Stochastic orderings between binomial and mixed binomial distributions

To avoid introduction of further notations, some of the notations of Section 3 will be repeated here in the context relevant to this section. For a fixed $p \in (0, 1)$ and a fixed positive integer n, let

N be a random variable having binomial distribution given by

$$\Pr(N=k)=\frac{n!}{k!(n-k)!}\,p^k(1-p)^{n-k},\quad k=0,1,2,\ldots,n. \tag{6}$$

Let M be a binomial random variable with n number of trials and the probability of success Θ, where Θ is a non-degenerate random variable having the probability density function $g(\theta)$, $0<\theta<1$. As in Section 3, although we have assumed Θ to be a continuous random variable, all the results of this section hold even when Θ is assumed to be an arbitrary discrete random variable. Moreover, the assumption that Θ is non-degenerate is not restrictive. We have

$$\Pr(M=k)=\int_0^1\frac{n!}{k!(n-k)!}\,\theta^k(1-\theta)^{n-k}g(\theta)\,\mathrm{d}\theta,\quad k=0,1,\ldots,n. \tag{7}$$

In this section, we make comparisons between random variables N and M, given by (6) and (7), respectively, with respect to the likelihood ratio, stochastic, uniform variability and expectation orderings. The following lemma will be useful in proving the main results:

Lemma 4.1. *Define*, $b_1(k)=E(\Theta^{k+1}(1-\Theta)^{n-k-1})/E(\Theta^k(1-\Theta)^{n-k-1})$, $k=0,1,\ldots,n-1$, $p_0=b_1(0)$, $p_1=1-[E((1-\Theta)^n)]^{1/n}$, $p_2=E(\Theta)$, $p_3=[E(\Theta^n)]^{1/n}$, $p_4=b_1(n-1)$, $s_k(x)=\sum_{j=0}^{k}n!/(j!(n-j)!)x^{(n-j)/n}(1-x^{1/n})^j$, $0<x<1$, $k=0,1,\ldots,n-1$ and $t_k(x)=\sum_{j=0}^{k}n!/(j!(n-j)!)x^{j/n}(1-x^{1/n})^{n-j}$, $0<x<1$, $k=0,1,\ldots,n-1$. *Then,*

(a) $b_1(k)$ *is an increasing function of* $k\in\{0,1,\ldots,n-1\}$,
(b) *for each fixed* $k\in\{0,1,\ldots,n-1\}$, $s_k(x)$ *is a concave function of* $x\in(0,1)$,
(c) *for each fixed* $k\in\{0,1,\ldots,n-1\}$, $t_k(x)$ *is a convex function of* $x\in(0,1)$,
(d) $0<p_0\leqslant p_1\leqslant p_2\leqslant p_3\leqslant p_4<1$.

Proof. (a) We may write

$$b_1(k)=E(V_k),\quad k=0,1,\ldots,n-1,$$

where V_k is a random variable having the pdf

$$\psi_k^*(x)=D_kx^k(1-x)^{n-k-1}g(x),\quad 0<x<1;$$

here D_k is the normalizing constant.

Fix $k\in\{0,1,\ldots,n-1\}$. Clearly, $\psi_{k+1}^*(x)/\psi_k^*(x)$ is an increasing function of $x\in(0,1)$, which implies that $V_{k+1}\geqslant_{\rm lr}V_k$, which further implies that $V_{k+1}\geqslant_{\rm st}V_k$ and therefore $b_1(k+1)=E(V_{k+1})\geqslant E(V_k)=b_1(k)$. Since the chosen $k\in\{0,1,\ldots,n-1\}$ was arbitrary, the assertion follows.

(b) For $k=0$, note that $s_0(x)=x$ is both concave and convex. For $k\in\{1,2,\ldots,n-1\}$, we have

$$\frac{\mathrm{d}}{\mathrm{d}x}s_k(x)=\frac{(n-1)!}{k!(n-k-1)!}\left(\frac{1-x^{1/n}}{x^{1/n}}\right)^k,\quad 0<x<1$$

which is clearly a decreasing function of $x\in(0,1)$. This proves the assertion (b).

(c) For $k \in \{0, 1, \dots, n-1\}$, we have

$$\frac{\mathrm{d}}{\mathrm{d}x} t_k(x) = -\frac{(n-1)!}{k!(n-k-1)!} \left(\frac{1-x^{1/n}}{x^{1/n}}\right)^{n-k-1}, \quad 0 < x < 1$$

which is clearly an increasing function of $x \in (0,1)$. This proves the assertion (c).

(d) $p_3 \leqslant p_4$ follows using the convexity of the function $e_1(x) = x^{n/(n-1)}$, $0 < x < 1$, and Jensen's inequality; $p_2 \leqslant p_3$ follows using the convexity of the function $e_2(x) = x^n$, $0 < x < 1$, and Jensen's inequality; and $p_1 \leqslant p_2$ follows using the convexity of the function $e_3(x) = (1-x)^n$, $0 < x < 1$, and Jensen's inequality. The inequality $p_0 \leqslant p_1$ follows from the inequality $p_3 \leqslant p_4$, with Θ replaced by $1 - \Theta$. □

The following theorem provides conditions on parameters p, p_0, p_1, p_2, p_3 and p_4 that ensure M is larger than N in various orderings.

Theorem 4.1. *Let N and M be random variables having distributions given by* (6) *and* (7), *respectively. Then, under the notations of Lemma* 4.1,

(a) $M \geqslant_{\mathrm{lr}} N$ *if and only if* $0 < p \leqslant p_0$,
(b) $N \geqslant_{\mathrm{lr}} M$ *if and only if* $p_4 \leqslant p < 1$,
(c) $M \geqslant_{\mathrm{st}} N$ *if and only if* $0 < p \leqslant p_1$,
(d) $N \geqslant_{\mathrm{st}} M$ *if and only if* $p_3 \leqslant p < 1$,
(e) $M \geqslant_{\mathrm{uv}} N$ *if and only if* $p_1 < p < p_3$,
(f) $M \geqslant_{\mathrm{E}} (\leqslant_{\mathrm{E}})\, N$ *if and only if* $0 < p \leqslant (\geqslant) p_2$.

Proof. (a) and (b). Consider the likelihood ratio

$$l(k) = \frac{\Pr(M=k)}{\Pr(N=k)} = \frac{E(\Theta^k(1-\Theta)^{n-k})}{p^k(1-p)^{n-k}}, \quad k = 0, 1, 2, \dots, n. \tag{8}$$

Then,

$$\begin{aligned} &l(k+1) \geqslant (\leqslant) l(k), \quad \forall k = 0, 1, \dots, n-1, \\ &\qquad \Leftrightarrow p \leqslant (\geqslant)\, b_1(k), \quad \forall k = 0, 1, \dots, n-1 \\ &\qquad \Leftrightarrow p \leqslant (\geqslant) p_0\ (p_4), \quad \text{(by virtue of Lemma 4.1 (a))} \end{aligned}$$

which provides a necessary and sufficient condition for the ratio in (8) to be non-decreasing (non-increasing). Hence assertions (a) and (b) follow.

(c) First, suppose that $0 < p \leqslant p_1$. For, $k = 0, 1, 2, \dots, n$, consider

$$\begin{aligned} \Delta_p^*(k) &= \Pr(N \leqslant k) - \Pr(M \leqslant k) \\ &= \sum_{j=0}^{k} \frac{n!}{j!(n-j)!}\, p^j (1-p)^{n-j} - \sum_{j=0}^{k} \frac{n!}{j!(n-j)!} E(\Theta^j (1-\Theta)^{n-j}). \end{aligned}$$

Since the binomial distribution is stochastically increasing in the success probability p, to show that $M \geqslant_{\rm st} N$, it is enough to establish that

$$\Delta^*_{p_1}(k) \geqslant 0, \quad \forall k = 0, 1, 2, \ldots, n.$$

For fixed $k \in \{0, 1, 2, \ldots, n\}$, we have

$$\Delta^*_{p_1}(k) = \sum_{j=0}^{k} \frac{n!}{j!(n-j)!}[1 - \{E((1-\Theta)^n)\}^{1/n}]^j [E(\{1-\Theta\}^n)]^{(n-j)/n}$$

$$- \sum_{j=0}^{k} \frac{n!}{j!(n-j)!} E(\Theta^j (1-\Theta)^{n-j}).$$

By applying Jensen's inequality to the function $s_k(x)$, defined in Lemma 4.1, it follows from above that

$$\Delta^*_{p_1}(k) \geqslant 0.$$

Conversely, suppose that $M \geqslant_{\rm st} N$. Then

$$\Pr(N \leqslant 0) \geqslant \Pr(M \leqslant 0)$$

$$\Leftrightarrow p \leqslant p_1.$$

Hence the assertion (c) is proved.

(d) First, suppose that $p_3 \leqslant p < 1$. As in (c), to show that $N \geqslant_{\rm st} M$, it is enough to establish that

$$\Delta^*_{p_3}(k) \leqslant 0, \quad \forall k = 0, 1, 2, \ldots, n.$$

For fixed $k \in \{0, 1, 2, \ldots, n\}$, we have

$$\Delta^*_{p_3}(k) = \sum_{j=0}^{k} \frac{n!}{j!(n-j)!}[E(\Theta^n)]^{j/n}[1 - \{E(\Theta^n)\}^{1/n}]^{n-j}$$

$$- \sum_{j=0}^{k} \frac{n!}{j!(n-j)!} E(\Theta^j (1-\Theta)^{n-j}).$$

By applying Jensen's inequality to the function $t_k(x)$, defined in Lemma 4.1, it follows from above that

$$\Delta^*_{p_3}(k) \leqslant 0.$$

Conversely, suppose that $N \geqslant_{\rm st} M$. Then

$$\Pr(N \geqslant n) \geqslant \Pr(M \geqslant n)$$

$$\Leftrightarrow p_3 \leqslant p < 1,$$

which proves the assertion (d).

(e) Suppose that $p_1 < p < p_3$. Then, it follows from (c) and (d) that random variables M and N are not ordered by the simple stochastic order. Also, it follows from the arguments used in the proof of (a) and (b) that $\Pr(N = k)/\Pr(M = k)$ is unimodal, implying that $M \geqslant_{uv} N$. The converse part follows using similar arguments.

(f) This is obvious. □

Remarks. 1. We note that $\Pr(M \geqslant 1) \geqslant \Pr(N \geqslant 1) \Leftrightarrow \Pr(M = 0) \leqslant \Pr(N = 0) \Leftrightarrow 0 < p \leqslant p_1 \Leftrightarrow M \geqslant_{st} N$, as proved in the above theorem. Thus, it is interesting to observe that $\Pr(M \geqslant 1) \geqslant \Pr(N \geqslant 1)$ ensures that $M \geqslant_{st} N$. Similarly, $\Pr(N = n) \geqslant \Pr(M = n)$ is enough to ensure that $N \geqslant_{st} M$.

2. Unlike in the case of Poisson distribution, in the binomial case, there exist values of success probability p ($p_3 \leqslant p < 1$) such that $N \geqslant_{st} M$.

3. Theorem 4.1. (c) (Theorem 4.1 (d)) provides a condition under which sampling from the mixture of binomial distributions (binomial distribution) is more favorable than the sampling from the binomial distribution (mixture of binomial distributions).

4. Interestingly, when $n = 1$ and the mixing distribution belongs to the beta family, it can be shown that $p \leqslant p_0$ ($p \geqslant p_4$) $\Leftrightarrow$ CV(N) $\geqslant (\leqslant)$ CV(M), where CV(.) denotes the coefficient of variation. Thus, in this case, $M \geqslant_{lr} N$ ($N \geqslant_{lr} M$) if and only if CV(N) $\geqslant (\leqslant)$ CV(M).

In the following theorem, we compare random variables N and M when $E(\Theta) = p$. In this case, N and M have the same mean i.e., $np = E[N] = E[M] = np_2$.

Theorem 4.3. *Suppose that* $p_2 = E(\Theta) = p$. *Then,*

(a) $M \geqslant_{uv} N$,
(b) $M \geqslant_{cx} N$, *which further implies that* $\text{Var}(N) \leqslant \text{Var}(M)$,
(c) $\Pr(N \geqslant 1) > \Pr(M \geqslant 1)$,
(d) *neither M* is *larger than N in the stochastic ordering nor N is larger than M in the stochastic ordering*.

Proof. (a) Follows from Theorem 4.1 (e), since $p_1 \leqslant p_2 \leqslant p_3$.

(b) Since $M \geqslant_{uv} N$ and $E(M) = E(N)$, it implies that $M \geqslant_{cx} N$. Thus $E(\phi(N)) \leqslant E(\phi(M))$ for all convex functions $\phi(.)$. Thus $E(M^2) \geqslant E(N^2)$, which implies that $\text{Var}(M) \geqslant \text{Var}(N)$, since M and N have the same means.

(c) It is easily seen that $\Pr(N = 0) \leqslant \Pr(M = 0)$, if $p_1 \leqslant p < 1$. The proof follows by observing that $p_2 \geqslant p_1$.

(d) From (c), it follows that M cannot be larger than N in the simple stochastic order. Since $p_2 \leqslant p_3$, the fact that N cannot be larger than M in the simple stochastic order follows from Theorem 4.1 (d). □

Remarks. 1. Results (a) and (b) of Theorem 4.3 are proved under a more general setting by Shaked (1980).

2. Theorem 4.3 (c) suggests that if the probability of success of the underlying binomial distribution is the same as the mean of the mixing distribution, then sampling from the binomial distribution is more favorable than sampling from the mixture of binomial distributions.

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