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# Time-dependent fields of a current-carrying wire

D V Redžić<sup>1</sup> and V Hnizdo<sup>2</sup>

<sup>1</sup> Faculty of Physics, University of Belgrade, PO Box 44, 11000 Beograd, Serbia

<sup>2</sup> National Institute for Occupational Safety and Health, Morgantown, WV 26505, USA

E-mail: [redzic@ff.bg.ac.rs](mailto:redzic@ff.bg.ac.rs)

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## Abstract

The electric and magnetic fields of an infinite straight wire carrying a steady current which is turned on abruptly are determined using Jefimenko's equations, starting from the standard assumption that the wire is electrically neutral in its rest frame. Some nontrivial aspects of the solution are discussed in detail.

## 1. Introduction

Consider an infinite straight linear wire carrying the current  $I(t) = 0$  for  $t \leq 0$ , and  $I(t) = I_0$  for  $t > 0$ . That is, a constant current  $I_0$  is turned on abruptly at time  $t = 0$ . What are the resulting electric and magnetic fields? This apparently simple electrodynamic problem is posed and solved as example 10.2 in Griffiths's excellent textbook [1]. Starting from the standard assumption that the wire is electrically neutral in its rest frame, with or without the current, which implies that the scalar potential  $V$  is zero, the retarded vector potential  $\mathbf{A}$  is calculated, and then the electric and magnetic fields are obtained according to  $\mathbf{E} = -\partial\mathbf{A}/\partial t$  and  $\mathbf{B} = \nabla \times \mathbf{A}$ , respectively. (Note that in this case the Coulomb and Lorenz gauges lead to the same potentials since the wire is electrically neutral.) While this solution is correct, we believe that the problem has some intriguing aspects and as such deserves further attention. In this paper we present a solution to the problem using Jefimenko's equations and point out some pitfalls which could be dangerous for novices. Thus, hopefully, our analysis should be instructive for advanced undergraduate and beginning graduate students.

## 2. Solution using retarded potentials

For the convenience of the reader, we first give the solution using retarded potentials, in more detail than in Griffiths's book.

As is well known, the retarded vector potential  $\mathbf{A}(\mathbf{r}, t)$  at field point  $\mathbf{r}$  and time  $t$  is given by

$$\mathbf{A}(\mathbf{r}, t) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{r}', t_r)}{|\mathbf{r} - \mathbf{r}'|} d^3r', \quad (1)$$

where  $\mathbf{J}(\mathbf{r}', t_r)$  is the current density at a source point  $\mathbf{r}'$  and the retarded time  $t_r = t - |\mathbf{r} - \mathbf{r}'|/c$ . Let the infinitely long wire lie along the  $z$  axis. The current in the wire, which is assumed to have an infinitesimal cross section, is turned on abruptly at  $t = 0$ , and thus the current density can be expressed as

$$\mathbf{J}(s, t) = \frac{I_0}{2\pi} \frac{\delta(s)}{s} \Theta(t) \hat{\mathbf{z}}, \quad (2)$$

where  $s$  is the distance from the wire,  $\delta(s)$  is the one-dimensional Dirac delta function normalized as  $\int_0^\infty \delta(s) ds = 1$  and  $\Theta(t)$  is the Heaviside step function,

$$\Theta(t) = \begin{cases} 0, & \text{if } t \leq 0 \\ 1, & \text{if } t > 0. \end{cases} \quad (3)$$

The setup is not realistic, but, in principle, it could be realized approximately with a large superconducting circular loop of negligible cross section in an inhomogeneous axially symmetric magnetic field, the symmetry axis coinciding with the axis of the loop. If the loop, initially at rest *with no current*, is moved quickly along the symmetry axis into a new resting position, a persistent current is produced in it, since the total magnetic flux through the loop is constant (see, e.g., [2]).

Equations (1) and (2) imply that the vector potential  $\mathbf{A}$  at a distance  $s$  from the wire is given by

$$\begin{aligned} \mathbf{A}(s, t) &= \frac{\mu_0}{4\pi} \hat{\mathbf{z}} \int_0^\infty s' ds' \int_0^{2\pi} d\phi' \int_{-\infty}^\infty dz' \frac{I_0}{2\pi} \frac{\delta(s')}{s'} \frac{\Theta(t - d/c)}{d} \\ &= \frac{\mu_0 I_0}{4\pi} \hat{\mathbf{z}} \int_{-\infty}^\infty \frac{\Theta(t - \sqrt{s^2 + z'^2}/c)}{\sqrt{s^2 + z'^2}} dz'. \end{aligned} \quad (4)$$

Here, cylindrical coordinates  $s, \phi, z$  are used and  $d = [s^2 + s'^2 - 2ss' \cos(\phi - \phi') + (z - z')^2]^{1/2}$ , which is the distance between the field point  $(s, \phi, z)$  and a source point  $(s', \phi', z')$ ; in the second line, the integration with respect to  $s'$  and a transformation  $z - z' \rightarrow z'$  reduce the distance to  $(s^2 + z'^2)^{1/2}$ . As demanded by the problem's symmetry, the vector potential is independent of  $z$  and  $\phi$ . The Heaviside-function factor in the integrand of the integral in the second line of (4) causes the potential to vanish at times  $t < s/c$  and limits the integration interval to the values of  $z'$  satisfying

$$|z'| \leq \sqrt{c^2 t^2 - s^2}. \quad (5)$$

Integral (4) for the vector potential  $\mathbf{A}$  thus evaluates as<sup>3</sup>

$$\begin{aligned} \mathbf{A}(s, t) &= \frac{\mu_0 I_0}{4\pi} \hat{\mathbf{z}} \Theta(t - s/c) \int_{-\sqrt{c^2 t^2 - s^2}}^{\sqrt{c^2 t^2 - s^2}} \frac{dz'}{\sqrt{s^2 + z'^2}} \\ &= \frac{\mu_0 I_0}{2\pi} \hat{\mathbf{z}} [\ln(ct + \sqrt{c^2 t^2 - s^2}) - \ln s] \Theta(t - s/c). \end{aligned} \quad (6)$$

The electric field is therefore given by

$$\begin{aligned} \mathbf{E}(s, t) &= -\partial \mathbf{A}(s, t) / \partial t \\ &= -\frac{\mu_0 I_0}{2\pi} \hat{\mathbf{z}} \frac{c}{\sqrt{c^2 t^2 - s^2}} \Theta(t - s/c) \end{aligned} \quad (7)$$

<sup>3</sup> The standard convention is understood according to which  $f(x)\Theta(x - x_0) = 0$  whenever  $x \leq x_0$ , even when the expression  $f(x)$  happens not to be defined at these values of  $x$ .

and the magnetic field by<sup>4</sup>

$$\begin{aligned}\mathbf{B}(s, t) &= \nabla \times \mathbf{A}(s, t) = -(\partial A_z / \partial s) \hat{\boldsymbol{\phi}} \\ &= \frac{\mu_0 I_0}{2\pi s} \hat{\boldsymbol{\phi}} \frac{ct}{\sqrt{c^2 t^2 - s^2}} \Theta(t - s/c).\end{aligned}\quad (8)$$

In both (7) and (8), the delta-function terms that arose from the derivatives of the Heaviside-step function in (6) dropped out on account of the property

$$f(x)\delta(x - x_0) = f(x_0)\delta(x - x_0) \quad (9)$$

of the delta function. Inspecting equations (7) and (8), we see that the fields  $\mathbf{E}(s, t)$  and  $\mathbf{B}(s, t)$  attain in the limit  $t \rightarrow \infty$  their familiar static values 0 and  $(\mu_0 I_0 / 2\pi s) \hat{\boldsymbol{\phi}}$ , respectively, and that both these fields diverge when  $t \rightarrow s/c$ .

### 3. Solution using Jefimenko's equations

As is now well known, starting from the retarded solution to the inhomogeneous wave equations for the fields  $\mathbf{E}$  and  $\mathbf{B}$  [4, 5], or from the familiar retarded potentials [1, 6], or otherwise [7], the time-dependent generalizations of the Coulomb and Biot–Savart laws can be derived:

$$\mathbf{E}(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0} \int \left[ \frac{\varrho(\mathbf{r}', t_r)}{\mathcal{R}^3} \mathcal{R} + \frac{\dot{\varrho}(\mathbf{r}', t_r)}{c\mathcal{R}^2} \mathcal{R} - \frac{\dot{\mathbf{J}}(\mathbf{r}', t_r)}{c^2\mathcal{R}} \right] d^3 r', \quad (10)$$

$$\mathbf{B}(\mathbf{r}, t) = \frac{\mu_0}{4\pi} \int \left[ \frac{\mathbf{J}(\mathbf{r}', t_r)}{\mathcal{R}^3} + \frac{\dot{\mathbf{J}}(\mathbf{r}', t_r)}{c\mathcal{R}^2} \right] \times \mathcal{R} d^3 r', \quad (11)$$

where  $\varrho$  is the volume charge density,  $\mathcal{R} \equiv \mathbf{r} - \mathbf{r}'$ , and the dots denote partial differentiation with respect to time. These equations, showing explicitly *true* sources of  $\mathbf{E}$  and  $\mathbf{B}$ , were first derived by Jefimenko [4]. We now shall calculate the fields  $\mathbf{E}$  and  $\mathbf{B}$  in the problem at hand using Jefimenko's equations.

In our problem, the charge density vanishes,

$$\varrho = 0, \quad (12)$$

since by assumption the wire is electrically neutral, and using equation (2) we get

$$\mathbf{J}(s', t_r) = \frac{I_0}{2\pi} \frac{\delta(s')}{s'} \Theta(t - \sqrt{s^2 + z'^2}/c) \hat{\mathbf{z}}, \quad (13)$$

$$\dot{\mathbf{J}}(s', t_r) = \frac{I_0}{2\pi} \Theta(t - s/c) \frac{\delta(s')}{s'} \delta(t - \sqrt{s^2 + z'^2}/c) \hat{\mathbf{z}}. \quad (14)$$

Here, cylindrical coordinates are used again, and the delta-function property (9) and the taking, with no loss of generality, the field coordinate  $z$  to be 0 reduced the retarded time  $t_r$  to the same value as that in equation (4); the step-function factor in (14) expresses the fact that the step function in (13) entails that not only the current density itself but also its partial time derivative vanishes for times  $t < s/c$ . Substitution into Jefimenko's equation (10) then gives

$$\mathbf{E}(s, t) = -\frac{I_0}{4\pi\epsilon_0 c^2} \hat{\mathbf{z}} \Theta(t - s/c) \int_0^\infty s' ds' \frac{\delta(s')}{s'} \int_{-\infty}^\infty dz' \frac{\delta(t - \sqrt{s^2 + z'^2}/c)}{\sqrt{s^2 + z'^2}}, \quad (15)$$

<sup>4</sup> It is perhaps worthwhile to note that expression (6) for the vector potential  $\mathbf{A}(s, t)$  resembles the quasi-static vector potential at a distance  $s$  from the midpoint of a straight wire of finite length  $2l = 2\sqrt{c^2 t^2 - s^2}$  carrying a constant current  $I_0$ . A calculation of the magnetic field according to  $\mathbf{B} = \nabla \times \mathbf{A}$  in which the distance-dependent length  $2l$  is treated as a constant would be equivalent to the use of the Biot–Savart law. However, this law is not applicable beyond the quasi-static regime (see, e.g., [3]).

which can be evaluated easily using the decomposition of the delta function [5, 8]

$$\delta(t - \sqrt{s^2 + z^2}/c) = \frac{c^2 t}{\sqrt{c^2 t^2 - s^2}} [\delta(z' - \sqrt{c^2 t^2 - s^2}) + \delta(z' + \sqrt{c^2 t^2 - s^2})] \quad (16)$$

as

$$\mathbf{E}(s, t) = -\frac{\mu_0 I_0}{2\pi} \hat{\mathbf{z}} \frac{c}{\sqrt{c^2 t^2 - s^2}} \Theta(t - s/c), \quad (17)$$

in full agreement with the electric field (7), obtained using the retarded vector potential. In a similar fashion, using equations (11), (13), (14) and (16) we obtain

$$\begin{aligned} \mathbf{B}(s, t) &= \frac{\mu_0 I_0}{4\pi} \Theta(t - s/c) s \hat{\boldsymbol{\phi}} \left[ \int_{-\sqrt{c^2 t^2 - s^2}}^{\sqrt{c^2 t^2 - s^2}} \frac{dz'}{(s^2 + z'^2)^{3/2}} + \frac{1}{c} \int_{-\infty}^{\infty} \frac{\delta(t - \sqrt{s^2 + z'^2}/c)}{s^2 + z'^2} dz' \right] \\ &= \frac{\mu_0 I_0}{2\pi s} \hat{\boldsymbol{\phi}} \frac{ct}{\sqrt{c^2 t^2 - s^2}} \Theta(t - s/c), \end{aligned} \quad (18)$$

in full agreement with the magnetic field (8), obtained using the retarded vector potential.

#### 4. Discussion

At first sight, the fact that the fields  $\mathbf{E}$  and  $\mathbf{B}$  obtained diverge when  $t \rightarrow s/c+$  while vanishing for  $t < s/c$  may seem disturbing. However, a closer examination reveals that, in the correct solution to the problem,  $\mathbf{E}$  and  $\mathbf{B}$  *must* tend to infinity when  $t \rightarrow s/c+$ . This is more transparent through the use of Jefimenko's equations (10) and (11) than the use of a retarded vector potential. It is clear from equations (13), (14) and (16) that the abrupt turning on of the current at  $t = 0$  necessarily yields an infinite time derivative of the current density, producing in the fields a cylindrical 'shock wave' that diverges at the time  $t = s/c$  at a distance  $s$  from the wire. Similar to the instructive example of Jackson of an abruptly turned-on electric dipole [9], the diverging fields  $\mathbf{E}$  and  $\mathbf{B}$  here are artefacts of the unphysical, instantaneous turn-on of the current<sup>5</sup>.

The divergences disappear if the current is not turned on abruptly, but is increased gradually during a short time interval  $\tau$ . As a simple example, assume that the current increases linearly from zero at  $t = 0$  to a steady non-zero value  $I_0$  at  $t \geq \tau$ , replacing accordingly expression (2) for the current density by

$$\mathbf{J}(s, t) = \frac{I_0}{2\pi} \frac{\delta(s)}{s} \left[ \frac{t}{\tau} \Theta(t) - \frac{t - \tau}{\tau} \Theta(t - \tau) \right] \hat{\mathbf{z}}. \quad (19)$$

Using Jefimenko's equations, the resulting fields are then obtained to be

$$\begin{aligned} \mathbf{E}_\tau(s, t) &= -\frac{\mu_0 I_0}{2\pi} \frac{\hat{\mathbf{z}}}{\tau} [\ln(ct/s + \sqrt{c^2 t^2 - s^2}/s) \Theta(t - s/c) \\ &\quad - \ln(c(t - \tau)/s + \sqrt{c^2 (t - \tau)^2 - s^2}/s) \Theta(t - \tau - s/c)] \end{aligned} \quad (20)$$

and

$$\mathbf{B}_\tau(s, t) = \frac{\mu_0 I_0}{2\pi c} \frac{\hat{\boldsymbol{\phi}}}{s\tau} [\sqrt{c^2 t^2 - s^2} \Theta(t - s/c) - \sqrt{c^2 (t - \tau)^2 - s^2} \Theta(t - \tau - s/c)]. \quad (21)$$

<sup>5</sup> We remind the reader that a similar situation is found in the well-known  $RC$ -circuit problem of charging a capacitor of capacitance  $C$  by connecting it instantaneously to a constant voltage  $V$  through a resistor of resistance  $R$ , assuming that the charge  $Q$  on the positive plate is zero at  $t = 0$ . The standard (tacit) assumption that the inductance  $L$  of the circuit is zero then leads to the equation  $V = Q/C + RI$ . In the unphysical setup of the problem (the abrupt closing of a circuit with  $L = 0$ ), the correct solution *must* satisfy the unphysical initial condition  $I(t = 0) = V/R$ , despite the fact that the current  $I$  vanishes for  $t < 0$ .

While the fields (20) and (21) are finite for any non-zero parameter  $\tau$ , their limits  $\tau \rightarrow 0$  can be shown easily to be the fields (7) and (8), respectively, that diverge when  $t \rightarrow s/c$ .

There is another query. The retarded vector potential and Jefimenko's equations are both derived under the assumption that the sources (charges and currents) are localized in a finite region of space, but our problem involves an infinitely long current-carrying wire. Therefore, the question of the validity of the solution found arises<sup>6</sup>. However, inspecting equation (6) we see that for any given finite  $s$  and  $t$  only a finite segment of the wire contributes to the retarded vector potential. Figuratively speaking, retardation makes the infinitely long wire finite. Still, one should check that the vector potential (6) satisfies the requisite inhomogeneous wave equation,

$$\nabla^2 \mathbf{A}(s, t) - \frac{1}{c^2} \frac{\partial^2 \mathbf{A}(s, t)}{\partial t^2} = -\frac{\mu_0 I_0}{2\pi} \frac{\delta(s)}{s} \Theta(t) \hat{\mathbf{z}}, \quad (22)$$

and that the fields (7) and (8) satisfy all Maxwell's equations.

Let us check first whether Maxwell's equations are satisfied. Straightforward calculations yield that the fields (7) and (8) are divergenceless, confirming the equations  $\nabla \cdot \mathbf{E} = \rho/\epsilon_0$ , where  $\rho = 0$ , and  $\nabla \cdot \mathbf{B} = 0$ . While a straightforward calculation of the curl of the electric field (7) confirms that the fields obtained satisfy Faraday's law, a similar calculation of the curl of the magnetic field (8) using the standard cylindrical-coordinate formula

$$\nabla \times F(s) \hat{\boldsymbol{\phi}} = \frac{1}{s} \frac{\partial(sF)}{\partial s} \hat{\mathbf{z}} \quad (23)$$

appears to yield only that  $\nabla \times \mathbf{B} = \epsilon_0 \mu_0 \partial \mathbf{E} / \partial t$ , instead of the full Ampère–Maxwell law, which reads in our case

$$\nabla \times \mathbf{B} = \frac{\mu_0 I_0}{2\pi} \frac{\delta(s)}{s} \Theta(t) \hat{\mathbf{z}} + \epsilon_0 \mu_0 \frac{\partial \mathbf{E}}{\partial t}. \quad (24)$$

Here, however, it is important to bear in mind that when the sources of a field are idealized point, line or surface distributions of charge and/or current, described by generalized functions such as the Dirac delta function, great care must be taken to employ in the field or potential differential equations generalized (distributional) derivatives instead of the usual (classical) ones<sup>7</sup>. Keeping this in mind, a careful examination of the differential operation on the magnetic field (8) implied by formula (23) reveals that it involves an expression that can be written as the Laplacian of the natural logarithm of the cylindrical coordinate  $s$ ,<sup>8</sup>

$$\frac{1}{s} \frac{\partial}{\partial s} \left[ s \left( \frac{1}{s} \right) \right] = \frac{1}{s} \frac{\partial}{\partial s} \left[ s \frac{\partial}{\partial s} (\ln s) \right] = \nabla^2 (\ln s). \quad (25)$$

In terms of the usual (classical) derivatives, this Laplacian vanishes for all  $s > 0$  and is not defined at  $s = 0$ , but in view of the fact that the current density involves the delta function we should use here the relation

$$\nabla^2 (\ln s) = \frac{\delta(s)}{s}. \quad (26)$$

Employing this relation in the evaluation of  $\nabla \times \mathbf{B}$ , it is found easily that the fields (7) and (8) now satisfy the full Ampère–Maxwell law (24).

<sup>6</sup> Recall that in electrostatics the standard solution to the Poisson equation is not generally valid for charge distributions extending to infinity (see, e.g., [10, 11]).

<sup>7</sup> For example, calculating the Laplacian of the potential  $1/r$  of a unit point charge using classical derivatives yields  $\nabla^2(1/r) = 0$  for all  $r > 0$ ; at  $r = 0$ , the classical Laplacian is simply not defined. In the well-known relation  $\nabla^2(1/r) = -4\pi\delta(\mathbf{r})$ , the Laplacian is in fact the generalized one, as it must be since this relation expresses equality of two generalized functions. To avoid confusion, some authors denote generalized differential operators by a bar, writing thus  $\bar{\nabla}^2(1/r) = -4\pi\delta(\mathbf{r})$  [12–14].

<sup>8</sup> This fact is perhaps more transparent when using  $\nabla \times \mathbf{B} = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$  and recognizing that for  $\mathbf{A}$  expressed by equation (6),  $\nabla^2(\ln s)$  appears explicitly in the calculation of  $\nabla^2 \mathbf{A}$ .

Since the relation (26) may appear to be novel to some readers, we give an informal proof of it using a limiting procedure in which  $\ln s$  is regularized as  $\ln \sqrt{s^2 + a^2}$  and the limit  $a \rightarrow 0$  is taken after integrating the product of  $\nabla^2 \ln \sqrt{s^2 + a^2}$  and a well-behaved 'test' function  $f(s)$ :

$$\lim_{a \rightarrow 0} \int_0^\infty \nabla^2 \ln \sqrt{s^2 + a^2} f(s) s \, ds = \lim_{a \rightarrow 0} \int_0^\infty \left[ \frac{2a^2}{(s^2 + a^2)^2} \right] f(s) s \, ds. \quad (27)$$

The expression in square brackets is the Laplacian of  $\ln \sqrt{s^2 + a^2}$ , which is now a well-behaved function of  $s$  for any  $a \neq 0$ ; its integral over the whole plane is independent of  $a$ , equalling  $2\pi$ . Splitting the integral into integrals over intervals  $0 \leq s \leq S$  and  $S \leq s < \infty$  so that the function  $f(s)$  can be expanded in the first interval in a Taylor series around  $s = 0$ , the limit (27) can be evaluated as

$$\begin{aligned} \lim_{a \rightarrow 0} \int_0^S \frac{2a^2}{(s^2 + a^2)^2} \sum_{n=0}^\infty \frac{f^{(n)}(0) s^n}{n!} s \, ds + \lim_{a \rightarrow 0} \left[ 2a^2 \int_S^\infty \frac{f(s)}{(s^2 + a^2)^2} s \, ds \right] \\ = \lim_{a \rightarrow 0} \left[ \frac{f(0)}{1 + a^2/S^2} + \left( a \operatorname{arctg} \frac{S}{a} - \frac{Sa^2}{S^2 + a^2} \right) f'(0) + O(a^2, a^2 \ln a) + \dots \right] \\ = f(0). \end{aligned} \quad (28)$$

The second limit in the first line vanished since the integral of  $f(s)s/(s^2 + a^2)^2$ , where  $f(s)$  is assumed to be integrable over the whole interval  $(0, \infty)$ , is guaranteed to converge to a value that remains finite when  $a \rightarrow 0$ . Having thus shown that

$$\lim_{a \rightarrow 0} \int_0^\infty \nabla^2 \ln \sqrt{s^2 + a^2} f(s) s \, ds = f(0), \quad (29)$$

where  $f(s)$  may be any well-behaved function of  $s$ , we can write

$$\lim_{a \rightarrow 0} \nabla^2 \ln \sqrt{s^2 + a^2} = \delta(s)/s, \quad (30)$$

which is the relation (26) with  $\nabla^2 (\ln s) \equiv \lim_{a \rightarrow 0} \nabla^2 \ln \sqrt{s^2 + a^2}$ .<sup>9</sup> (Note that the relation (26) may be interpreted as the Poisson equation for the electrostatic potential  $-(1/2\pi\epsilon_0) \ln s$  of a charge density  $\delta(s)/2\pi s$ , which is that of an infinitely long straight line of charge of unit line charge density [5, 10, 11, 15].)

Checking whether the vector potential (6) satisfies the inhomogeneous wave equation (22) is a somewhat cumbersome but in every step straightforward calculation. Using the relation (26) and making extensive use of relation (9) in the terms containing  $\delta(t - s/c)$ , which arise through differentiations of  $\Theta(t - s/c)$ , we obtain

$$\nabla^2 \mathbf{A}(s, t) - \frac{1}{c^2} \frac{\partial^2 \mathbf{A}(s, t)}{\partial t^2} = -\frac{\mu_0 I_0}{2\pi} \frac{\delta(s)}{s} \Theta(t - s/c) \hat{\mathbf{z}}, \quad (31)$$

which confirms, as it must<sup>10</sup>, that the vector potential (6) satisfies the wave equation (22) since, on account of (9),  $\delta(s)\Theta(t - s/c) = \delta(s)\Theta(t)$ .

Closing the discussion, we remark that the solution to the problem of finding the fields due to an infinite straight linear wire carrying a constant current that is turned on abruptly appears to be relevant to the related problem of finding charges and fields in a current-carrying wire of finite cross section, where it appears that an infinite time is needed to establish a stationary charge distribution [16]. However, it should be noted that the time-dependent fields obtained

<sup>9</sup> Strictly speaking, the limit here and in (30) is the weak limit (see, e.g., [8]) and the Laplacian in (26) is the generalized (distributional) one (see, e.g., [12]).

<sup>10</sup> The facts that  $\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \epsilon_0 \mu_0 \partial \mathbf{E} / \partial t$ ,  $\mathbf{B} = \nabla \times \mathbf{A}$ ,  $\mathbf{E} = -\partial \mathbf{A} / \partial t$  and  $\nabla \cdot \mathbf{A} = 0$  already imply that  $\square \mathbf{A} = -\mu_0 \mathbf{J}$ .

here approach their values at  $t \rightarrow \infty$  very rapidly. For example, the field (18) at a distance  $s = 1$  m and a time  $t = 3 \mu\text{s}$  differs from its asymptotic value by less than one part in a million (the wire is then required to be at least 2 km long).

## 5. Conclusions

We calculated the electric and magnetic fields of an infinitely long wire carrying a constant current that is turned on abruptly, using Jefimenko's equations. Our calculations confirmed the brief solution given to this problem in an example of Griffiths's text, but our method appears to provide more insight than the standard approach via the retarded potentials: the divergence of the resulting fields  $\mathbf{E}(s, t)$  and  $\mathbf{B}(s, t)$  when  $t \rightarrow s/c$  can then be seen easily as a necessary consequence of the 'unphysical' setup of the problem. We also calculated the fields for a more realistic case in which the current in the wire is increased linearly in a nonzero time to its final steady value. We believe that our analysis was an instructive demonstration of not only the power but also the possible pitfalls of using the delta function in the calculations of electrostatics.

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