



A note on recovering the distributions from exponential moments



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ABSTRACT

The problem of recovering a cumulative distribution function of a positive random variable via the scaled Laplace transform inversion is studied. The uniform upper bound of proposed approximation is derived. The approximation of a compound Poisson distribution as well as the estimation of a distribution function of the summands given the sample from a compound Poisson distribution are investigated. Applying the simulation study, the question of selecting the optimal scaling parameter of the proposed Laplace transform inversion is considered. The behavior of the approximants are demonstrated via plots and table.

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1. Introduction

In risk theory it is often required to evaluate the compound (aggregated) distribution of severity losses when the number of claims during some period of time is random. Unfortunately, the closed form solutions for many such cases are not available. See for example, Gzyl and Tagliani [1] and the references therein. Also, the problem of decomposing the random sums represents another interesting and difficult probabilistic inverse problem. On the other hand, in many models of risk theory, one can evaluate or estimate the Laplace transform (or exponential moments) of the target distribution. Hence, by inverting the Laplace transform numerically we will be able to recover the underlying distribution. Also, the derivation of the rate of approximations of functions will contribute to the theory of approximations as well.

Recently, the so-called moment-recovered (MR) approximation of the Laplace transform inversion was suggested in [2]. The present note highlights additional property for the scaled version (2) of this approximation. The approximation of the Laplace transform inversion in the aforementioned work is mainly recommended for use in the framework of the Hausdorff moment problem, when the support of the target function F is a compact ($\text{supp}\{F\} = (0, T), T < \infty$). Here we suggest the modified, scaled version of the MR-Laplace transform inversion that enables us to apply it in the case of the Stieltjes moment problem as well, i.e., when $T = \infty$. The reader is referred to [3–6], where the questions on the moment-determinacy of probability distributions and their approximations in the framework of inverse moment problem are investigated. See also Tagliani and Velasquez [7], where the fractional moments are used to approximate the Laplace transform inversion. Regarding the conditions on the moment determinacy of the distributions of compound geometric sums we refer to Lin and Stoyanov [8] and the references therein.

There are several very well known techniques for calculation of the compound distributions, e.g., Panjer recursion, Fourier transform technique, shifted gamma approach (see, for example, [9]), and maximum entropy method using the fractional exponential moments in [1], among others. Recently, Buchmann and Grübel [10] proposed the estimator of the individual loss distribution which is based on the inversion of the Panjer recursion formula. Based on the reversion of the power series,

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the authors in [11] derived the weak convergence of the inverse estimator of the distribution of summands to a gaussian process. Very interesting results are derived in [12], where a model with a noisy Laplace transform is investigated. In this type of model the regularization technique is applied.

The main aim of this article is to derive the upper bound for MR-approximation $F_{\alpha,b}$ in the sup-norm when the underlying distribution F has unbounded support in \mathbb{R}_+ . Here, we applied our technique for recovery of a compound Poisson distribution as well as for estimation of the distribution of the summands (the individual claim sizes) of a random sum given the sample from the distribution of aggregated sums. It is worth noting that the MR-construction in (2) can be used not for only compound Poisson case but for other compound distribution as well. Also it is worth noting that the results of the current paper are easily extended to the multivariate case. This question will be studied in the forthcoming paper.

The article is organized as follows. In Section 2 the construction of the MR-approximation of the scaled Laplace transform inversion is introduced, and the uniform rate of approximation is established. In Section 3 we applied our construction to the problem of recovering the compound Poisson distribution as well as in decompounding the Poisson distribution. Several examples are considered as well. Based on simulation study, the graphical illustrations and table with the values of estimated approximation error are provided.

2. The rate of approximation

Assume that the distribution F is absolutely continuous and has a support $\mathbb{R}_+ = [0, \infty)$. Let f be its probability density function (pdf) with respect to the Lebesgue measure on \mathbb{R}_+ . In [2] we derived the Laplace transform inversion based on the moment-recovered approximation of the distributions in the Hausdorff moment problem. This inversion works well for distributions with a light tails. In this section we modify the aforementioned construction and study its behavior in the cases of a heavy tail distributions, e.g., a gamma and a log-normal.

Suppose that a random variable X distributed according to F . Assume also that we are given the sequence $\mu(F) = \{\mu_t(F), t \in \mathbb{N}_\alpha\}$ defined by the values of the scaled Laplace transform of F :

$$\mu_t(F) := \mathcal{L}_{F,b}(t) = \int_{\mathbb{R}_+} e^{-ctx} dF(x) \quad \text{for } t \in \mathbb{N}_\alpha = \{0, 1, \dots, \alpha\}, \alpha = 0, 1, \dots \quad (1)$$

To simplify the notations let us assume in (1) that the scale value $c = \ln b$ for some $1 < b \leq \exp(1)$. The problem of the optimal choice of the parameter b represents another question addressed in this article.

To approximate the cdf F , one can apply the result from Mnatsakanov [2]. Namely, let us introduce the scaled MR-Laplace transform inversion:

$$F_{\alpha,b}(x) := \bar{\mathcal{L}}_{\alpha,b}^{-1}(\mu(F))(x) = 1 - \sum_{k=0}^{\lfloor \alpha e^{-x \ln b} \rfloor} \sum_{j=k}^{\alpha} \binom{\alpha}{j} \binom{j}{k} (-1)^{j-k} \mu_j(F), \quad x \in \mathbb{R}_+. \quad (2)$$

Our approximation is based on applying the following relationship:

$$B_\alpha(u, v) = \sum_{k=0}^{\lfloor \alpha v \rfloor} \binom{\alpha}{k} u^k (1-u)^{\alpha-k} \rightarrow \begin{cases} 1, & u < v \\ 0, & u > v \end{cases}, \quad \text{as } \alpha \rightarrow \infty.$$

Let us denote the pdf of the $Beta(c, d)$ distribution by

$$\beta(u, c, d) = \frac{\Gamma(c+d)}{\Gamma(c)\Gamma(d)} u^{c-1} (1-u)^{d-1}, \quad 0 < u < 1, \quad (3)$$

where the shape parameters $c, d > 0$. Also for the simplicity of notations, let us write $\beta_{\alpha,x}(\cdot) := \beta(\cdot, c, d)$ when $c = \lfloor \alpha b^{-x} \rfloor + 1$ and $d = \alpha - \lfloor \alpha b^{-x} \rfloor + 1$. To approximate the survival function $S = 1 - F$ let us consider $S_{\alpha,b} = 1 - F_{\alpha,b}$. Let us denote by $\|\phi\|$ the sup-norm of a function $\phi: \mathbb{R}_+ \rightarrow \mathbb{R}$, and assume that for some $b \in (1, e]$:

$$M_k = \sup_{x \in \mathbb{R}_+} |f(x) b^{kx}|, \quad k = 1, 2 \quad \text{with } M_2 < \infty, \quad \text{and } M_3 = \sup_{x \in \mathbb{R}_+} |f'(x) b^{2x}| < \infty. \quad (4)$$

The following statement is true:

Theorem 1. If the functions f and f' are bounded on \mathbb{R}_+ and conditions (4) are satisfied, then $F_{\alpha,b}$ converges uniformly to F , and

$$\|F_{\alpha,b} - F\| \leq \frac{1}{\alpha+1} \left\{ \frac{M_1}{\ln^2 b} + \frac{M_2}{2 \ln b} + \frac{M_3}{2 \ln^2 b} \right\} + o\left(\frac{1}{\alpha}\right) \quad \text{as } \alpha \rightarrow \infty.$$

Proof. The proof repeats the steps used in the proof of Theorem 2 in [13]. Namely, first of all note that from (1) and (2) we have

$$\begin{aligned}
S_{\alpha,b}(x) &= 1 - F_{\alpha,b}(x) = \int_0^\infty \sum_{k=0}^{\lfloor \alpha b^{-x} \rfloor} \sum_{j=k}^{\alpha} \binom{\alpha}{j} \binom{j}{k} (b^{-t})^k (-b^{-t})^{j-k} dF(t) = \int_0^\infty \sum_{k=0}^{\lfloor \alpha b^{-x} \rfloor} \binom{\alpha}{k} (b^{-t})^k \sum_{m=0}^{\alpha-k} \binom{\alpha-k}{m} (-b^{-t})^m dF(t) \\
&= \int_0^\infty B_\alpha(b^{-t}, b^{-x}) dF(t) = - \int_0^1 B_\alpha(u, v) dG(u),
\end{aligned} \tag{5}$$

where $G(u) = F(-\log_b u)$ and $v = b^{-x}$. On the other hand, taking the derivative of

$$B_\alpha(u, v) = (1-u)^\alpha + \sum_{k=1}^{\lfloor \alpha v \rfloor} \binom{\alpha}{k} u^k (1-u)^{\alpha-k},$$

with respect to u we obtain

$$\begin{aligned}
B'_\alpha(u, v) &= -\alpha(1-u)^{\alpha-1} + \sum_{k=1}^{\lfloor \alpha v \rfloor} \binom{\alpha}{k} u^{k-1} (1-u)^{\alpha-k} - \sum_{k=1}^{\lfloor \alpha v \rfloor} \binom{\alpha}{k} u^k (\alpha-k)(1-u)^{\alpha-k-1} \\
&= -\alpha \binom{\alpha-1}{\lfloor \alpha v \rfloor} u^{\lfloor \alpha v \rfloor} (1-u)^{\alpha-1-\lfloor \alpha v \rfloor} = -\beta(u, c, d),
\end{aligned} \tag{6}$$

with $c = \lfloor \alpha v \rfloor + 1$ and $d = \alpha - \lfloor \alpha v \rfloor$. Hence, applying the integration by parts in the last integral of (5) and taking into account (6), where $v = b^{-x}$, we derive:

$$S_{\alpha,b}(x) = 1 - \int_0^1 G(u) \beta_{\alpha,x}(u) du = \int_0^1 \bar{G}(u) \beta_{\alpha,x}(u) du.$$

Therefore,

$$S_{\alpha,b}(x) - S(x) = \int_0^1 [\bar{G}(u) - \bar{G}(v)] \beta_{\alpha,x}(u) du, \tag{7}$$

where $\bar{G}(u) = 1 - G(u) = S(-\log_b u)$ and $\bar{G}(v) = S(x)$. It is worth mentioning that for the first two derivatives of \bar{G} :

$$g(u) = \frac{f(-\log_b u)}{u \ln b} \quad \text{and} \quad g'(u) = -\frac{f(-\log_b u)}{u^2 \ln b} - \frac{f'(-\log_b u)}{u^2 \ln^2 b},$$

we have:

$$\sup_{u \in [0,1]} g(u) = \frac{M_1}{\ln b} \quad \text{and} \quad \sup_{u \in [0,1]} |g'(u)| \leq \frac{M_2}{\ln b} + \frac{M_3}{\ln^2 b}, \tag{8}$$

respectively.

Note also that the mean and variance of the *Beta*(c, d) distribution defined in (3) are such that

$$\eta_\alpha := \int_0^1 u \beta_{\alpha,x}(u) du = \frac{\lfloor \alpha b^{-x} \rfloor + 1}{\alpha + 1}, \tag{9}$$

$$\sigma_\alpha^2 := \int_0^1 (u - \eta_\alpha)^2 \beta_{\alpha,x}(u) du = \frac{(\lfloor \alpha b^{-x} \rfloor + 1)(\alpha - \lfloor \alpha b^{-x} \rfloor)}{(\alpha + 1)^2 (\alpha + 2)} < \frac{1}{\alpha + 1}, \tag{10}$$

and

$$|\eta_\alpha - b^{-x}| \leq \frac{1}{(\alpha + 1) \ln b}. \tag{11}$$

Now, let us use the following notations $u \wedge v = \min(u, v)$ and $u \vee v = \max(u, v)$. Substitution of

$$\bar{G}(u) - \bar{G}(v) = g(v)(u - v) + \int_{u \wedge v}^{u \vee v} g'(s)(u \vee v - s) ds,$$

into (7) and taking into account (8)–(11) yields

$$\begin{aligned}
|S_{\alpha,b}(x) - S(x)| &= \left| \int_0^1 \beta_{\alpha,x}(u) \{ \bar{G}(u) - \bar{G}(b^{-x}) \} du \right| \leq \frac{M_1}{\ln b} |\eta_\alpha - b^{-x}| + \left(\frac{M_2}{\ln b} + \frac{M_3}{\ln^2 b} \right) \left\{ \frac{\sigma_\alpha^2}{2} + \frac{(\eta_\alpha - b^{-x})^2}{2} \right\} \\
&\leq \frac{1}{\alpha + 1} \left\{ \frac{M_1}{\ln^2 b} + \frac{M_2}{2 \ln b} + \frac{M_3}{2 \ln^2 b} \right\} + o\left(\frac{1}{\alpha}\right),
\end{aligned}$$

as $\alpha \rightarrow \infty$. \square

Remark 1. To approximate the probability density function $f = F'$ let us consider the ratio

$$\frac{\Delta F_{\alpha,b}(x_j)}{\Delta x_j} \quad \text{for } x_{j-1} \leq x < x_j,$$

where $\Delta F_{\alpha,b}(x_j) = F_{\alpha,b}(x_j) - F_{\alpha,b}(x_{j-1})$ and $x_j = (\ln \alpha - \ln(\alpha - j + 1)) / \ln b, j = 1, \dots, \alpha$. After a simple algebra and scaling this ratio by $(\alpha + 1)/\alpha$, one can derive the following approximation of f :

$$f_{\alpha,b}(x) = \frac{[\alpha b^{-x}] \ln(b) \Gamma(\alpha + 2)}{\alpha \Gamma([\alpha b^{-x}] + 1)} \sum_{m=0}^{\alpha - [\alpha b^{-x}]} \frac{(-1)^m \mu_{m+[\alpha b^{-x}]}(F)}{m! (\alpha - [\alpha b^{-x}] - m)!}, \quad x \in \mathbb{R}_+,$$

(cf. with the construction $\mathcal{L}_\alpha^{-1}v$ introduced in [2] when $b = \exp(1)$). The properties of $f_{\alpha,b}$ and its extended version to the bivariate case will be studied in the forthcoming paper. Below, see Fig. 1, the curves of $f_{\alpha,b}$ for two different distributions, Exp (β) and Gamma (a, β) are provided for two different values of b : $b = 1.08$ and $b = 1.15$. Here, we considered the rates $\beta = 0.2, 0.5$. Namely, we assume $X \sim \text{Exp}(0.2)$ in Fig. 1(a) and (b), and $X \sim \text{Gamma}(3, 0.5)$ in Fig. 1(c), respectively.

3. Some applications and examples

In many practical situations it is impossible to evaluate the exponential moments $\mu(F)$. For example, the log-normal distribution does not have an finite analytical form for its Laplace transform. In such cases the estimated $\hat{\mu}(F) = \{\hat{\mu}_j(F), j \in \mathbb{N}_\alpha\}$ exponential moments of F can be used in (2). This provides the MR-estimate of the Laplace inversion:

$$\hat{F}_{\alpha,b} := \tilde{\mathcal{L}}_{\alpha,b}^{-1} \hat{\mu}(F). \quad (12)$$

Note that if F is observed directly by means of a sample of i.i.d. random variables X_1, \dots, X_n , then (12) with the empirical exponential moments $\hat{\mu}(F)$ provides the estimate of the survival function S :

$$\hat{S}_{\alpha,b}(x) = 1 - \tilde{\mathcal{L}}_{\alpha,b}^{-1} \hat{\mu}(F)(x) = \int_0^\infty B_\alpha(b^{-t}, b^{-x}) d\hat{F}_n(t) = \frac{1}{n} \sum_{i=1}^n B_\alpha(b^{-X_i}, b^{-x}). \quad (13)$$

By \hat{F}_n in (13) we denote the empirical cdf of the sample X_1, \dots, X_n (cf. with the last line of equation (5)).

In Figs. 2–6 below, for the approximated and estimated curves of the underlying cdfs we use the green and blue colors, respectively. Note also that to make the approximation smoother one can linearize the step function by connecting the distinct values of $F_{\alpha,b}(x)$ at $x \in \{(\ln \alpha - \ln(\alpha - j + 1)) / \ln b, j = 1, \dots, \alpha\}$ via lines. See, for example the curve of corresponding version of $\hat{F}_{\alpha,b}$ in Fig. 2 (c).

To choose the optimal b one can use the simulated data-set. Namely, for each given α and n , let us calculate the average of $\|\hat{F}_{\alpha,b} - F\|$ over R replications:

$$d(\hat{F}_{\alpha,b}, F) = \frac{1}{R} \sum_{r=1}^R \|\hat{F}_{\alpha,b}^r - F\|. \quad (14)$$

After this step, define the optimal $b^* = \arg\min_{1 < b \leq e} d(\hat{F}_{\alpha,b}, F)$. Here, in (14), $\hat{F}_{\alpha,b}^r$ represents the value of $\hat{F}_{\alpha,b}$ obtained on the r -th replication, $1 \leq r \leq R$.

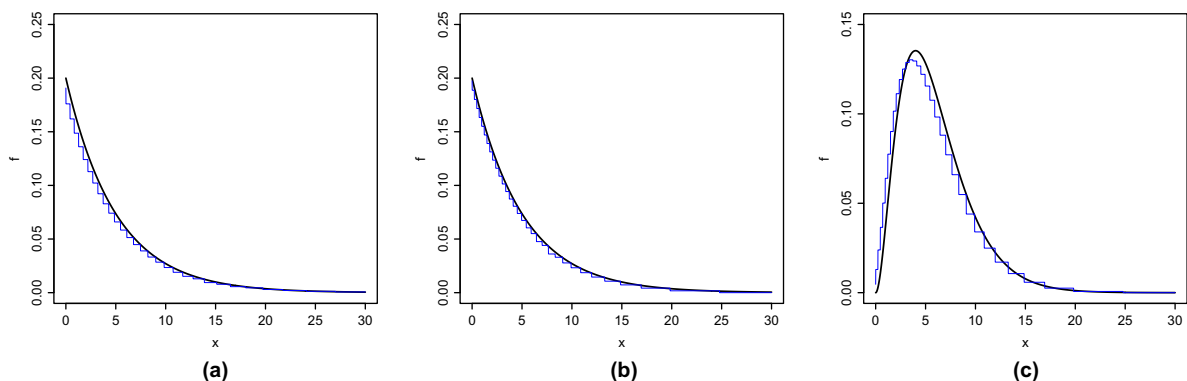


Fig. 1. (a) Approximation of Exp (0.2) pdf by $f_{\alpha,b}$ when $\alpha = 32$ and (a) $b = 1.08$; (b) $b = 1.15$; (c) Approximation of Gamma (3, 0.5) pdf by $f_{\alpha,b}$ when $\alpha = 32$ and $b = 1.15$.

Table 1The values of $d(\widehat{F}_{\alpha,b}, F)$ and b^* (in brackets) for Gamma distributions.

Model	n/α	20	25	32
Gamma (4, 2.5)	200	0.0928 (1.60)	0.0813 (1.70)	0.0710 (1.85)
	500	0.0885 (1.70)	0.0738 (1.70)	0.0666 (1.80)
	10^3	0.0875 (1.55)	0.0740 (1.70)	0.0640 (1.70)
Gamma (4, 0.4)	200	0.0918 (1.06)	0.0854 (1.075)	0.0707 (1.06)
	500	0.0881 (1.06)	0.0757 (1.06)	0.0560 (1.06)
	10^3	0.0846 (1.06)	0.0714 (1.06)	0.0548 (1.06)

In the case of two different gamma distributions, we recorded the values of $d(\widehat{F}_{\alpha,b}, F)$ when $\alpha = 20, 25, 32$, and the sample size $n = 100k, k = 2, 5, 10$. See Table 1 below, where, we took the number of replications $R = 50$ and $b = 1 + j\Delta b, j = 1, \dots, 80$, with $\Delta b = 0.025$. The simulations show that, for given α and n , the value of optimal b^* depends on the mean of F .

Example 1. Assume that X follows the exponential distribution with the rate $\beta, X \sim \text{Exp}(\beta)$. In Fig. 2 (a) and (b) and Fig. 2 (c), we plotted the curves of $F_{\alpha,b}, \widehat{F}_{\alpha,b}$, and corresponding cdf F when $\beta = 0.5$ and 0.10 , respectively.

Example 2. Let $X \sim \text{Gamma}(\text{shape} = a, \text{rate} = \beta)$. Consider two cases: $(a, \beta) \in \{(4, 2.5), (4, 0.4)\}$. The plots in Fig. 3 (a) and (b) provide the approximated and estimated curves $F_{\alpha,b}$ and $\widehat{F}_{\alpha,b}$, respectively, when $\alpha = 32, b = 1.7$, and $n = 500$ in the first case with $\beta = 2.5$. Fig. 3(c) displays both approximated and estimated curves when $\alpha = 32, b = 1.1$, and $n = 500$ in the second case.

Example 3. Let $X \sim \text{Log-normal}(\mu, \sigma)$. Consider again two cases with $(\mu, \sigma) \in \{(0, 1), (1, 2)\}$. See Fig. 4 (a) and (b), where the curves of the target cdfs and their estimated counterparts are displayed.

From Examples 2–4 we see that the optimal value of scaling parameter b is a decreasing function of the mean of X .

Recovery of a Compound Poisson distribution. Let X_1, X_2, \dots be i.i.d. random variables from cdf F defined on \mathbb{R}_+ . Consider a random sum

$$Y = X_1 + \dots + X_N. \quad (15)$$

Here the number of summands N is a discrete random variable which is assumed to be independent from the summands. For example, in insurance literature, the aggregated claim size Y often follows a compound Poisson distribution, i.e., when the number of claims N has a Poisson distribution with some intensity $\lambda > 0$. Assuming that λ is given, we would like to approximate or estimate the distribution of aggregated claim sizes G when F is known or unknown, respectively. One can use the relationship

$$G = \sum_{j=0}^{\infty} p_j F^{*j}, \quad \text{with } p_j = P(N = j), \quad (16)$$

to derive the Laplace transform of the aggregated claim sizes and then to approximate its inversion via (2) (cf. with Panjer [14]).

Assume first that the distribution F of the individual claim sizes is known and N has a Poisson distribution, $N \sim \text{Pois}(\lambda)$, with some known intensity $\lambda > 0$. From the relationship (16) we have:

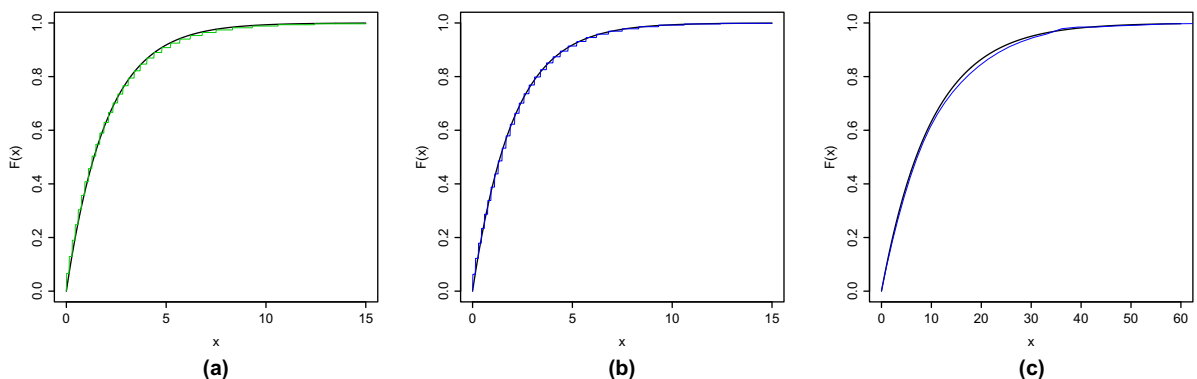


Fig. 2. (a) Approximation of $\text{Exp}(0.5)$ cdf by $F_{\alpha,b}$ when $\alpha = 32, b = 1.25$; (b) Estimation of $\text{Exp}(0.5)$ cdf by $\widehat{F}_{\alpha,b}$ when $\alpha = 32, b = 1.25$, and $n = 500$; (c) Estimation of $\text{Exp}(0.10)$ cdf by smoothed version of $\widehat{F}_{\alpha,b}$ when $\alpha = 32, b = 1.03$, and $n = 500$.

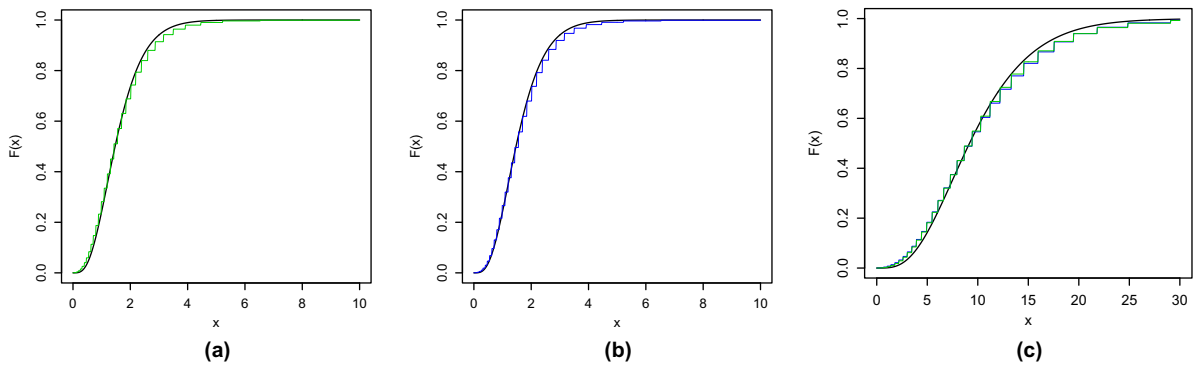


Fig. 3. (a) Approximation of Gamma (4, 2.5) cdf by $F_{x,b}$ when $\alpha = 32$, $b = 1.7$; (b) Estimation of Gamma (4, 2.5) cdf by $\hat{F}_{x,b}$ when $\alpha = 32$, $b = 1.7$, and $n = 500$; (c) Approximation and Estimation of Gamma (4, 0.4) cdf by $F_{x,b}$ and $\hat{F}_{x,b}$ when $\alpha = 32$, $b = 1.1$, and $n = 500$, respectively

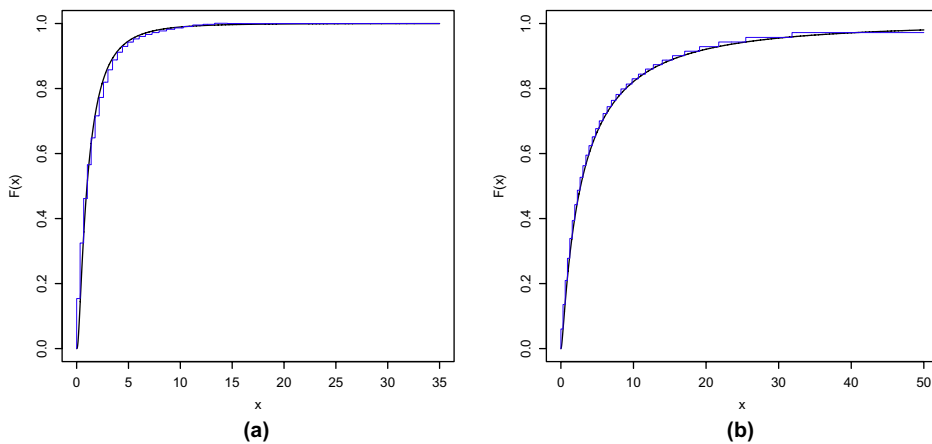


Fig. 4. (a) Estimation of Log-normal (0,1) distribution by $\hat{F}_{x,b}$ when $\alpha = 32$, $b = 1.25$, and $n = 200$; (b) Estimation of Log-normal (1,2) distribution by $\hat{F}_{x,b}$ when $\alpha = 32$, $b = 1.115$ and $n = 1000$.

$$\mu_t(G) = \mathcal{L}_{G,b}(t) = e^{\lambda[\mathcal{L}_{F,b}(t)-1]}, \quad t \geq 0. \quad (17)$$

Now let us apply (2), where the exponential moment sequence of the target distribution is $\mu(G) = \{\mu_j(G), j \in \mathbb{N}_\alpha\}$ with $\mu_j(G) = \mathcal{L}_{G,b}(j)$. This yields the scaled Laplace transform inversion for recovering G :

$$G_{\alpha,b}(x) := \tilde{\mathcal{L}}_{\alpha,b}^{-1}\mu(G)(x) = 1 - \sum_{k=0}^{\lfloor \alpha e^{-x \ln b} \rfloor} \sum_{j=k}^{\alpha} \binom{\alpha}{j} \binom{j}{k} (-1)^{j-k} \mu_j(G), \quad x \in \mathbb{R}_+. \quad (18)$$

Example 4. Let $X \sim \text{Gamma}(a, \beta)$, with $a = 2$, $\beta = 2$ and $\lambda = 4$. Fig. 5 (a) provides the curve $G_{\alpha,b}$ approximating the Compound Poisson cdf G when $\alpha = 32$ and $b = 1.115$. For comparison we also plotted the curve of the empirical cdf of the sample Y_1, \dots, Y_m drawn from G with the sample size $m = 10^4$. Note that in this example, the scaled Laplace transform of G has a very simple form:

$$\mathcal{L}_{G,b}(t) = e^{\lambda \left[\left(\frac{\beta}{\beta + t \ln b} \right)^a - 1 \right]}.$$

Now assume that F is unknown but the sample X_1, \dots, X_n from F is available. In this case, one can substitute the empirical counterpart $\hat{\mathcal{L}}_{F,b}$ of $\mathcal{L}_{F,b}$ into (17).

We simulated $n = 800$ observations from Gamma(2, 2) distribution, and applied (18) with the estimated version of $\mu_j(G)$:

$$\hat{\mu}_j(G) := \hat{\mathcal{L}}_{G,b}(j) = e^{\lambda [\hat{\mathcal{L}}_{F,b}(j) - 1]}.$$

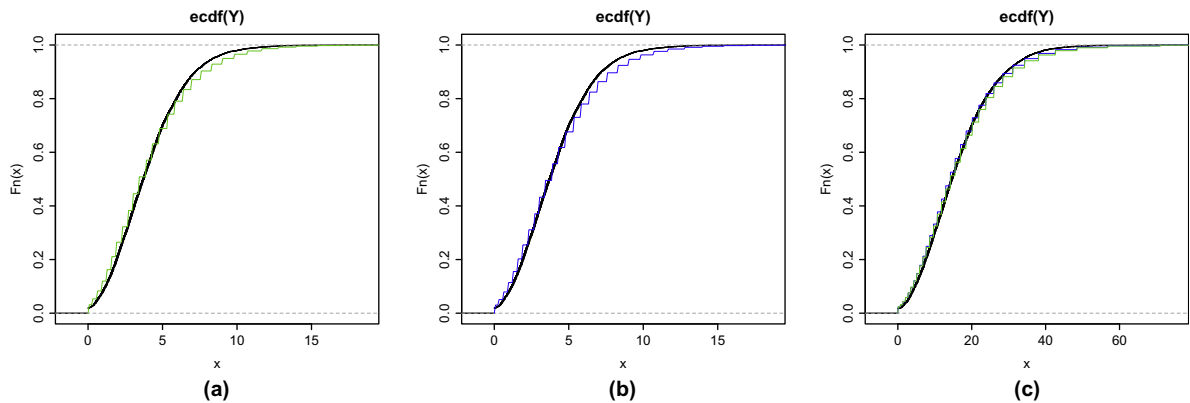


Fig. 5. (a) Approximation of Compound Poisson distribution G by $G_{\alpha,b}$, when $X \sim \text{Gamma}(2,2)$, and $\lambda = 4$, $\alpha = 32$, $b = 1.115$; (b) Estimation of G by $\hat{G}_{\alpha,b}$ with $n = 800$ and $\lambda = 4$, $\alpha = 32$, $b = 1.115$; and (c) Approximation and Estimation of G by $G_{\alpha,b}$ and $\hat{G}_{\alpha,b}$, when $X \sim \text{Gamma}(2,0.5)$, $n = 800$, $\lambda = 4$, $\alpha = 32$, and $b = 1.05$.

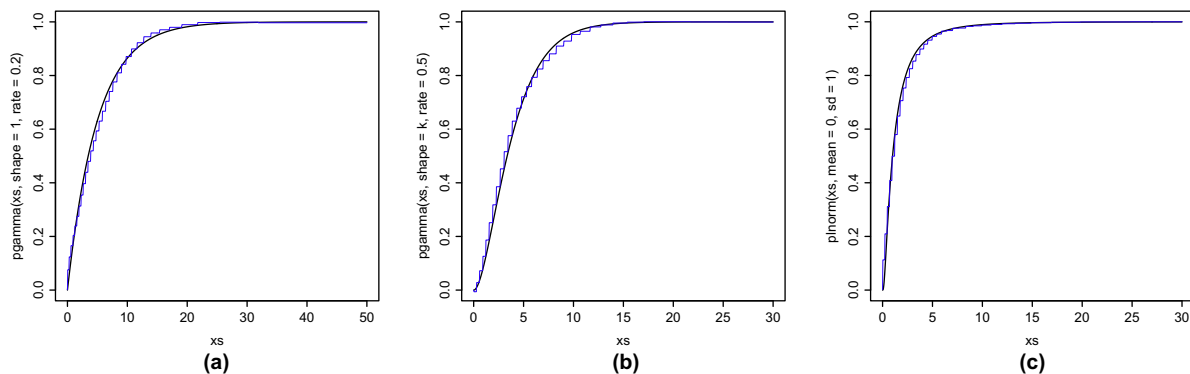


Fig. 6. (a) Estimation of individual claim size distribution F by $\hat{F}_{\alpha,b}$ with $\alpha = 32$, $b = 1.115$, $n = 10^3$, when (a) $X \sim \text{Exp}(0.2)$ and $\lambda = 4$; (b) $X \sim \text{Gamma}(2,0.5)$ and $\lambda = 2$; and (c) $X \sim \text{Log-normal}(0,1)$ and $\lambda = 2$.

As a result, we derived the estimate of the compound Poisson distribution:

$$\hat{G}_{\alpha,b}(x) = \bar{\mathcal{L}}_{\alpha,b}^{-1} \hat{\mu}(G)(x), \quad x \in \mathbb{R}_+.$$

Fig. 5 (b) displays the estimator of G based on $\hat{G}_{\alpha,b}$. Fig. 5 (c) displays the approximant of G and corresponding estimator $\hat{G}_{\alpha,b}$ when $X_i \sim \text{Gamma}(2,0.5)$ with $n = 800$, $\lambda = 4$, and $b = 1.05$. To make the comparison, in all three plots of Fig. 5 the empirical cdf (the black curve) of the sample from cdf G with the sample size $m = 10^4$ is displayed as well. Again, we see that when the mean of G is increasing the optimal value of parameter b is decreasing.

Decomposing a Poisson distribution. Now let us apply the MR-approach in the following inverse problem when, given the distribution G in (16) or the sample from G , we would like to determine the distribution F or estimate it, respectively. We will call this problem decomposing.

Different approaches were proposed to handle the problem of decomposing. See, for example, Buchmann and Grübel [10], and Bogsted and Pitts [11] among others.

Assume now that the exponential moments $\mu_j(F)$ of unknown cdf F are recovered somehow. Then to recover cdf F one can apply the Laplace transform inversion (2). In particular, when $N \sim \text{Poisson}(\lambda)$, the relationship (17) gives

$$\mu_j(F) = \frac{1}{\lambda} \ln(\mu_j(G)) + 1. \quad (19)$$

Hence, to estimate the distribution of the summands F , we can estimate its exponential moments $\mu_j(F)$ via substitution of the corresponding exponential moments of the empirical distribution \hat{G}_n into (19), and then apply (2). This yields the MR-estimate of F :

$$\hat{F}_{\alpha,b} = \bar{\mathcal{L}}_{\alpha,b}^{-1} \hat{\mu}(F), \quad (20)$$

where $\hat{\mu}(F) = \{\hat{\mu}_j(F), j \in \mathbb{N}_x\}$ and $\hat{\mu}_j(F) = \frac{1}{\lambda} \ln(\mu_j(\hat{G}_n)) + 1$.

Finally, in Fig. 6, we plotted the estimates of the individual claim sizes for three different models: Exp (0.2), Gamma (2, 0.5), and Log-normal (0,1), given the samples Y_1, \dots, Y_n of size $n = 1000$ from the compound Poisson distribution G with parameter $\lambda = 2, 4$. In all three cases we took $b = 1.115$.

4. Conclusions

We derived the uniform upper bound for the rate of MR-approximation of a cdf F supported by a positive half line. In the case when the mean of the underlying distribution is not very large, the proposed modification of the moment-recovered Laplace transform inversion $F_{\alpha,b}$, with $1 < b \leq \exp(1)$, is recommended rather than the one when the scaling parameter $b = \exp(1)$ (cf. Mnatsakanov [2]). The main advantage of MR-approximation $F_{\alpha,b}$ is its easiness of implementation. The disadvantage of $F_{\alpha,b}$ is that it becomes a constant beyond the point $\ln \alpha / \ln b$. Hence, when α is not large enough, we recommend a choice of b very close to 1.

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