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LETTERS AND COMMENTS

Comment on 'Electromagnetic force on a moving dipole'

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Abstract
Using the Lagrangian formalism, the force on a moving dipole derived by Kholmetskii *et al* (2011 *Eur. J. Phys.* **32** 873–81) is found to be missing some important terms.

Recently, Kholmetskii *et al* [1] have inspected the analysis of Vekstein [2] of the force on a small system of zero net charge but with electric and magnetic dipole moments $\boldsymbol{d}$ and $\boldsymbol{m}$, respectively, moving with velocity $\boldsymbol{v}$ in an electromagnetic field $\boldsymbol{E}$, $\boldsymbol{B}$. They concluded that the last term in Vekstein's expression for the force,

$$\boldsymbol{F}_{\mathrm{V}} = (\boldsymbol{d}\cdot\nabla)\boldsymbol{E} + \nabla(\boldsymbol{m}\cdot\boldsymbol{B}) + \frac{1}{c}\,\boldsymbol{d}\times(\boldsymbol{v}\cdot\nabla)\boldsymbol{B}, \tag{1}$$

is erroneous and derived for the force the following expression:

$$\boldsymbol{F}_{\mathrm{KMY}} = \nabla(\boldsymbol{d}\cdot\boldsymbol{E}) + \nabla(\boldsymbol{m}\cdot\boldsymbol{B}) + \frac{1}{c}\frac{\partial}{\partial t}(\boldsymbol{d}\times\boldsymbol{B}). \tag{2}$$

In both (1) and (2), the dipole moments $\boldsymbol{d}$ and $\boldsymbol{m}$ may arise from the rest-frame dipole moments $\boldsymbol{m}_0$ and $\boldsymbol{d}_0$ according to transformations that read to first order in v/c as [3][1]

$$\boldsymbol{d} = \boldsymbol{d}_0 + \frac{1}{c}\,\boldsymbol{v}\times\boldsymbol{m}_0, \qquad \boldsymbol{m} = \boldsymbol{m}_0 - \frac{1}{c}\,\boldsymbol{v}\times\boldsymbol{d}_0. \tag{3}$$

In this comment, we employ the Lagrangian formalism to obtain the force on a zero-charge particle with rest-frame electric and magnetic dipole moments, moving in a static electromagnetic field. The resulting force, correct to first order in v/c, is

$$\boldsymbol{F} = \nabla(\boldsymbol{d}\cdot\boldsymbol{E}) + \nabla(\boldsymbol{m}\cdot\boldsymbol{B}) - \frac{1}{c}\,\boldsymbol{m}_0\times(\boldsymbol{v}\cdot\nabla)\boldsymbol{E} + \frac{1}{c}\,\boldsymbol{d}_0\times(\boldsymbol{v}\cdot\nabla)\boldsymbol{B}$$
$$- \frac{1}{c}\,\dot{\boldsymbol{m}}_0\times\boldsymbol{E} + \frac{1}{c}\,\dot{\boldsymbol{d}}_0\times\boldsymbol{B}. \tag{4}$$

[1] The magnetic moment $\boldsymbol{m} = -\boldsymbol{v}\times\boldsymbol{d}_0/c$ of a moving rest-frame electric dipole $\boldsymbol{d}_0$ is not the standard magnetic dipole moment $(1/2c)\int \mathrm{d}^3r\,\boldsymbol{r}\times\boldsymbol{J}(\boldsymbol{r})$ of a divergenceless current distribution $\boldsymbol{J}(\boldsymbol{r})$. It is to be regarded as a quantity the correct use of which will yield the force on the moving dipole exerted by an external magnetic field.

Here, the term $(1/c)\, \boldsymbol{d}_0 \times (\boldsymbol{v} \cdot \nabla)\boldsymbol{B}$ agrees to first order in v/c with the last term in Vekstein's force (1), disputed in [1], but the term's 'dual', $-(1/c)\, \boldsymbol{m}_0 \times (\boldsymbol{v} \cdot \nabla)\boldsymbol{E}$, absent in both (1) and (2), is present, too; the last two terms, where the dots denote time derivatives, are due to the possible time dependence of the dipole moments $\boldsymbol{m}_0$ and $\boldsymbol{d}_0$. Note also that $\nabla(\boldsymbol{d} \cdot \boldsymbol{E}) = (\boldsymbol{d} \cdot \nabla)\boldsymbol{E}$ when the field $\boldsymbol{E}$ is static ($\nabla \times \boldsymbol{E} = 0$).

The moving particle acquires moments (3), and thus its interaction with a static field $\boldsymbol{E}$, $\boldsymbol{B}$ is

$$\begin{aligned} U &= -\boldsymbol{d} \cdot \boldsymbol{E} - \boldsymbol{m} \cdot \boldsymbol{B} \\ &= -\boldsymbol{d}_0 \cdot \boldsymbol{E} - \frac{1}{c}(\boldsymbol{v} \times \boldsymbol{m}_0) \cdot \boldsymbol{E} - \boldsymbol{m}_0 \cdot \boldsymbol{B} + \frac{1}{c}(\boldsymbol{v} \times \boldsymbol{d}_0) \cdot \boldsymbol{B}. \end{aligned} \tag{5}$$

The nonrelativistic Lagrangian of such a particle is then

$$\begin{aligned} L &= \frac{1}{2} m v^2 - U \\ &= \frac{1}{2} m v^2 + (\boldsymbol{d}_0 + \boldsymbol{v} \times \boldsymbol{m}_0/c) \cdot \boldsymbol{E} + (\boldsymbol{m}_0 - \boldsymbol{v} \times \boldsymbol{d}_0/c) \cdot \boldsymbol{B} \\ &= \frac{1}{2} m v^2 + \boldsymbol{d}_0 \cdot \boldsymbol{E} + \boldsymbol{m}_0 \cdot \boldsymbol{B} + \frac{1}{c} \boldsymbol{v} \cdot (\boldsymbol{m}_0 \times \boldsymbol{E} - \boldsymbol{d}_0 \times \boldsymbol{B}), \end{aligned} \tag{6}$$

yielding a canonical momentum

$$\frac{\partial L}{\partial \boldsymbol{v}} = m\boldsymbol{v} + \frac{1}{c}(\boldsymbol{m}_0 \times \boldsymbol{E} - \boldsymbol{d}_0 \times \boldsymbol{B}). \tag{7}$$

The use of the Lagrange equation

$$\frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial L}{\partial \boldsymbol{v}} = \frac{\partial L}{\partial \boldsymbol{r}} \tag{8}$$

with the time derivative

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{m}_0 \times \boldsymbol{E} - \boldsymbol{d}_0 \times \boldsymbol{B}) &= \boldsymbol{m}_0 \times (\boldsymbol{v} \cdot \nabla)\boldsymbol{E} + \dot{\boldsymbol{m}}_0 \times \boldsymbol{E} \\ &\quad - \boldsymbol{d}_0 \times (\boldsymbol{v} \cdot \nabla)\boldsymbol{B} - \dot{\boldsymbol{d}}_0 \times \boldsymbol{B} \end{aligned} \tag{9}$$

results in a force

$$\begin{aligned} m\dot{\boldsymbol{v}} &= \nabla[(\boldsymbol{d}_0 + \boldsymbol{v} \times \boldsymbol{m}_0/c) \cdot \boldsymbol{E}] + \nabla[(\boldsymbol{m}_0 - \boldsymbol{v} \times \boldsymbol{d}_0/c) \cdot \boldsymbol{B}] \\ &\quad - \frac{1}{c} \boldsymbol{m}_0 \times (\boldsymbol{v} \cdot \nabla)\boldsymbol{E} + \frac{1}{c} \boldsymbol{d}_0 \times (\boldsymbol{v} \cdot \nabla)\boldsymbol{B} - \frac{1}{c} \dot{\boldsymbol{m}}_0 \times \boldsymbol{E} + \frac{1}{c} \dot{\boldsymbol{d}}_0 \times \boldsymbol{B}, \end{aligned} \tag{10}$$

which, recalling transformations (3), is force (4), understood as mass times acceleration.

Let us now apply expression (4) to simplifying examples. Consider first a constant electric dipole $\boldsymbol{d}_0$ with vanishing magnetic moment $\boldsymbol{m}_0$, moving with velocity $\boldsymbol{v}$ in a static magnetic field $\boldsymbol{B}$ and zero electric field $\boldsymbol{E}$. Modelling the dipole as charges q and $-q$ separated by a small displacement $\boldsymbol{l}$, so that $\boldsymbol{d}_0 = q\boldsymbol{l}$, the force on the dipole is the net Lorentz force on these charges, which, since $\boldsymbol{B}(\boldsymbol{r} + \boldsymbol{l}) - \boldsymbol{B}(\boldsymbol{r}) \approx (\boldsymbol{l} \cdot \nabla)\boldsymbol{B}(\boldsymbol{r})$, is given by

$$\boldsymbol{F}_{\mathrm{L}} = \frac{1}{c} \boldsymbol{v} \times (\boldsymbol{d}_0 \cdot \nabla)\boldsymbol{B}. \tag{11}$$

But according to expression (4) with $\boldsymbol{E} = 0$ and $\dot{\boldsymbol{d}}_0 = 0$, the force on the dipole is given as

$$\begin{aligned} \boldsymbol{F} &= \nabla(\boldsymbol{m} \cdot \boldsymbol{B}) + \frac{1}{c} \boldsymbol{d}_0 \times (\boldsymbol{v} \cdot \nabla)\boldsymbol{B} \\ &= -\frac{1}{c} \nabla[(\boldsymbol{v} \times \boldsymbol{d}_0) \cdot \boldsymbol{B}] + \frac{1}{c} \boldsymbol{d}_0 \times (\boldsymbol{v} \cdot \nabla)\boldsymbol{B}, \end{aligned} \tag{12}$$

where we used $\boldsymbol{m} = -\boldsymbol{v} \times \boldsymbol{d}_0/c$. Let us check whether that agrees with force (11). Using standard vector calculus identities, we have

$$
\begin{aligned}
-\nabla[(\boldsymbol{v} \times \boldsymbol{d}_0) \cdot \boldsymbol{B}] &= \nabla[\boldsymbol{d}_0 \cdot (\boldsymbol{v} \times \boldsymbol{B})] \\
&= (\boldsymbol{d}_0 \cdot \nabla)(\boldsymbol{v} \times \boldsymbol{B}) + \boldsymbol{d}_0 \times [\nabla \times (\boldsymbol{v} \times \boldsymbol{B})] \\
&= \boldsymbol{v} \times (\boldsymbol{d}_0 \cdot \nabla)\boldsymbol{B} - \boldsymbol{d}_0 \times (\boldsymbol{v} \cdot \nabla)\boldsymbol{B}.
\end{aligned} \tag{13}
$$

Substituting (13) into (12) yields the Lorentz force (11). The term $(1/c)\boldsymbol{d}_0 \times (\boldsymbol{v} \cdot \nabla)\boldsymbol{B}$, called into question in [1], is thus needed in expression (4) to yield the correct force (11).

Consider further a constant magnetic dipole $\boldsymbol{m}_0$ with vanishing electric moment $\boldsymbol{d}_0$, moving with velocity $\boldsymbol{v}$ in a static electric field $\boldsymbol{E}$ and zero magnetic field $\boldsymbol{B}$. According to expression (4) with $\boldsymbol{B} = 0$ and $\dot{\boldsymbol{m}}_0 = 0$, the force on the dipole is now given by

$$
\begin{aligned}
\boldsymbol{F} &= \nabla(\boldsymbol{d} \cdot \boldsymbol{E}) - \frac{1}{c}\,\boldsymbol{m}_0 \times (\boldsymbol{v} \cdot \nabla)\boldsymbol{E} \\
&= \frac{1}{c}\,\nabla[(\boldsymbol{v} \times \boldsymbol{m}_0) \cdot \boldsymbol{E}] - \frac{1}{c}\,\boldsymbol{m}_0 \times (\boldsymbol{v} \cdot \nabla)\boldsymbol{E},
\end{aligned} \tag{14}
$$

where we used $\boldsymbol{d} = \boldsymbol{v} \times \boldsymbol{m}_0/c$. Using standard identities, we have

$$
\begin{aligned}
\nabla[(\boldsymbol{v} \times \boldsymbol{m}_0) \cdot \boldsymbol{E}] &= -\nabla[\boldsymbol{m}_0 \cdot (\boldsymbol{v} \times \boldsymbol{E})] \\
&= -(\boldsymbol{m}_0 \cdot \nabla)(\boldsymbol{v} \times \boldsymbol{E}) - \boldsymbol{m}_0 \times [\nabla \times (\boldsymbol{v} \times \boldsymbol{E})] \\
&= -(\boldsymbol{m}_0 \cdot \nabla)(\boldsymbol{v} \times \boldsymbol{E}) - \boldsymbol{m}_0 \times [(\nabla \cdot \boldsymbol{E})\boldsymbol{v} - (\boldsymbol{v} \cdot \nabla)\boldsymbol{E}].
\end{aligned} \tag{15}
$$

Substituting (15) with $\nabla \cdot \boldsymbol{E} = 4\pi\rho = 0$ (the dipole is assumed to be moving in a region where the external charge density ρ vanishes) into (14) yields

$$
\boldsymbol{F} = -\frac{1}{c}\,(\boldsymbol{m}_0 \cdot \nabla)(\boldsymbol{v} \times \boldsymbol{E}). \tag{16}
$$

This is the force on a magnetic dipole moving in a static electric field that is used in an analysis by Aharonov *et al* [4] of the Aharonov–Casher effect [5], which is the electric analogue of the well-known Aharonov–Bohm effect. In that analysis, the force is obtained by invoking the so-called hidden momentum of a magnetic dipole (see e.g. [6] and references therein), but we obtained it here directly using the transformation $\boldsymbol{d} = \boldsymbol{v} \times \boldsymbol{m}_0/c$ and the Lagrange formalism (a similar procedure has been used in [7]). Without the term $-(1/c)\boldsymbol{m}_0 \times (\boldsymbol{v} \cdot \nabla)\boldsymbol{E}$ in expression (4), the generally accepted quantum-mechanical nature of the Aharonov–Casher effect could be questioned [8].

Interestingly, Schwinger scattering [9], which is the scattering of neutrons by the electric field of an atomic nucleus, can be shown to be due to force (16). The Schwinger-scattering Hamiltonian,

$$
H = -\frac{\hbar^2}{2m}\,\nabla^2 + \frac{\mathrm{i}\mu\hbar}{mc}\,\boldsymbol{\sigma} \cdot (\boldsymbol{E} \times \nabla) \tag{17}
$$

(see [10], equation (42.1)), is transformed into a classical Hamiltonian by the replacements $-\mathrm{i}\hbar\nabla \to \boldsymbol{P}$ and $\mu\boldsymbol{\sigma} \to \boldsymbol{m}_0$, where $\boldsymbol{P}$ and $\boldsymbol{m}_0$ are the classical canonical momentum and magnetic dipole moment, respectively:

$$
\begin{aligned}
H &= \frac{1}{2m}\,P^2 - \frac{1}{mc}\,\boldsymbol{m}_0 \cdot (\boldsymbol{E} \times \boldsymbol{P}) \\
&= \frac{1}{2m}\,P^2 - \frac{1}{mc}\,\boldsymbol{P} \cdot (\boldsymbol{m}_0 \times \boldsymbol{E}).
\end{aligned} \tag{18}
$$

If this Hamiltonian can be derived from the Lagrangian

$$
L = \frac{1}{2}\,mv^2 + \frac{1}{c}\,\boldsymbol{v} \cdot (\boldsymbol{m}_0 \times \boldsymbol{E}), \tag{19}
$$

which is that of equation (6) with $\boldsymbol{d}_0 = 0$ and $\boldsymbol{B} = 0$ and thus it yields force (16) when $\dot{\boldsymbol{m}}_0 = 0$, then the force implied by Hamiltonian (18) is the same, at least to order v/c. But Lagrangian (19) yields a canonical momentum

$$\boldsymbol{P} = \frac{\partial L}{\partial \boldsymbol{v}} = m\boldsymbol{v} + \frac{1}{c}\,(\boldsymbol{m}_0 \times \boldsymbol{E}), \tag{20}$$

and thus the Lagrange transformation [11] results in a Hamiltonian

$$\begin{aligned} H = \boldsymbol{P}\cdot\boldsymbol{v} - L &= \frac{1}{2} m v^2 \\ &= \frac{1}{2m}\,(\boldsymbol{P} - \boldsymbol{m}_0 \times \boldsymbol{E}/c)^2 \\ &\approx \frac{1}{2m}\,P^2 - \frac{1}{mc}\,\boldsymbol{P}\cdot(\boldsymbol{m}_0 \times \boldsymbol{E}), \end{aligned} \tag{21}$$

dropping in the last line a term proportional to $1/c^2$ in accordance with the fact that the Schwinger Hamiltonian (17) is nonrelativistic. The last line of (21) is indeed Hamiltonian (18).

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