

Potentials of a uniformly moving point charge in the Coulomb gauge*

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Abstract

The Coulomb-gauge vector potential of a uniformly moving point charge is obtained by calculating the gauge function for the transformation between the Lorenz and Coulomb gauges. The expression obtained for the difference between the vector potentials in the two gauges is shown to satisfy a Poisson equation to which the inhomogeneous wave equation for this quantity can be reduced. The right-hand side of the Poisson equation involves an important but easily overlooked delta-function term that arises from a second-order partial derivative of the Coulomb potential of a point charge.

1. Introduction

Gauge invariance is an important property of electrodynamics. Notwithstanding the thorough attention it has received in textbooks [1–4], the topic of gauge invariance seems to be still very much alive. Recently, articles have appeared on the resolution of apparent causality problems in the Coulomb gauge [5], the transformation from the Lorenz gauge to the Coulomb and some other gauges [6], the Coulomb-gauge vector potential in terms of the magnetic field [7], and the historical development of the whole concept of gauge invariance [8]. A discordant voice in this is a claim by Onoochin [9] that the electric field of a uniformly moving point charge comes out differently when it is calculated in the Lorenz and Coulomb gauges, which that author takes as evidence that the two gauges are not physically equivalent.

Curiously, it is difficult to find in the literature an explicit expression for the Coulomb-gauge vector potential of a uniformly moving charge. To the present author's knowledge, such an expression has appeared only in a short comment [10], where it is obtained directly from the well-known electric field and Coulomb-gauge scalar potential of the charge. The formulae of Jackson [6] for the transformation between the Lorenz and Coulomb gauges thus have come timely to provide an analytical check on the claim of Onoochin. The purpose of the present paper is to clear up the problem of the Coulomb-gauge vector potential

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of a uniformly moving charge using Jackson's results as well as the method employed by Onoouchin, in which the difference between the Coulomb- and Lorenz-gauge vector potentials in this problem is to be found by solving a Poisson equation.

In section 2, we obtain the difference between the potentials in the Lorenz and Coulomb gauges for a uniformly moving point charge using the formalism of Jackson's paper [6], which guarantees that the two gauges will yield the same electric and magnetic fields. In section 3, we demonstrate that the expression obtained for the difference between the vector potentials in the two gauges satisfies the inhomogeneous wave equation for this difference, and in section 4 we show where and why the procedure used by Onoouchin for solving that wave equation by reducing it to a Poisson equation went wrong. It will turn out that the right-hand side of this equation must include an easily overlooked delta-function term to yield the correct solution. Concluding remarks are made in section 5, and an appendix contains some calculational details.

2. Transformation from the Lorenz gauge to the Coulomb gauge

The scalar potential V and the vector potential $\mathbf{A}$ of a uniformly moving point charge in the Lorenz gauge, defined by the condition $\nabla \cdot \mathbf{A} + \partial \mathbf{A} / c \partial t = 0$, are well known. For a point charge q moving with a constant velocity $\mathbf{v} = v\hat{\mathbf{x}}$, the Lorenz-gauge potentials V_L and $\mathbf{A}_L$ are given by (we shall use the Gaussian system of units):

$$V_L(\mathbf{r}, t) = \frac{q}{\sqrt{(x - vt)^2 + \frac{1}{\gamma^2}(y^2 + z^2)}} \quad \mathbf{A}_L(\mathbf{r}, t) = \frac{\mathbf{v}}{c} V_L(\mathbf{r}, t) \quad (1)$$

where $\gamma = (1 - v^2/c^2)^{-1/2}$ and, for simplicity, the charge is assumed to pass through the origin $x=y=z=0$ at time $t = 0$ (see, e.g., [3], section 19-3). In the Coulomb gauge, defined by the condition $\nabla \cdot \mathbf{A} = 0$, the scalar potential V_C of the charge takes a particularly simple form,

$$V_C(\mathbf{r}, t) = \frac{q}{\sqrt{(x - vt)^2 + y^2 + z^2}} \quad (2)$$

as the scalar potential in this gauge is exactly the same as that of the instantaneous Coulomb interaction of electrostatics. On the other hand, the Coulomb-gauge vector potential $\mathbf{A}_C$ is the retarded solution to a relatively complicated inhomogeneous wave equation

$$\square \mathbf{A}_C = -\frac{4\pi}{c} q \mathbf{v} \delta(\mathbf{r} - \mathbf{v}t) + \nabla \frac{\partial V_C}{c \partial t} \quad (3)$$

where $\square = \nabla^2 - \partial^2 / c^2 \partial t^2$ is the d'Alembertian operator and $q \mathbf{v} \delta(\mathbf{r} - \mathbf{v}t)$ is the point-charge current density of the present problem (see, e.g., [1], section 6.3).

The gauge invariance of electrodynamics implies that there is a gauge function χ_C that connects the Coulomb- and Lorenz-gauge potentials by

$$V_C = V_L - \frac{\partial \chi_C}{c \partial t} \quad \mathbf{A}_C = \mathbf{A}_L + \nabla \chi_C \quad (4)$$

which ensures that the Lorenz-gauge and Coulomb-gauge potentials will yield the same electric and magnetic fields. This is because the fields are generated from the potentials via the prescription

$$\mathbf{E}(\mathbf{r}, t) = -\nabla V(\mathbf{r}, t) - \frac{\partial \mathbf{A}(\mathbf{r}, t)}{c \partial t} \quad \mathbf{B}(\mathbf{r}, t) = \nabla \times \mathbf{A}(\mathbf{r}, t) \quad (5)$$

and thus any electric-field and magnetic-field differences that could arise from the use of the different gauges are guaranteed to vanish:

$$-\nabla(V_C - V_L) - \frac{\partial(\mathbf{A}_C - \mathbf{A}_L)}{c \partial t} = \nabla \frac{\partial \chi_C}{c \partial t} - \frac{\partial}{c \partial t} \nabla \chi_C = 0 \quad (6)$$

$$\nabla \times (\mathbf{A}_C - \mathbf{A}_L) = \nabla \times \nabla \chi_C = 0. \quad (7)$$

Nevertheless, it should be instructive to demonstrate explicitly that it is indeed the case also in the present problem by finding the requisite gauge function. Before we turn to this task, we give here for completeness the fields that the prescription (5) yields with the Lorenz-gauge potentials V_L and $\mathbf{A}_L$ of equation (1):

$$\mathbf{E}(\mathbf{r}, t) = q \frac{\gamma(\mathbf{r} - \mathbf{v}t)}{[\gamma^2(x - vt)^2 + y^2 + z^2]^{3/2}} \quad \mathbf{B}(\mathbf{r}, t) = \frac{1}{c} \mathbf{v} \times \mathbf{E}(\mathbf{r}, t). \quad (8)$$

The same expressions for the electric and magnetic fields can be obtained also by Lorentz-transforming the electrostatic Coulomb field of the charge from its rest frame to the ‘laboratory’ frame.

Jackson [6] has derived the following integral expression for the gauge function χ_C in terms of the charge density:

$$\chi_C(\mathbf{r}, t) = -c \int d^3r' \frac{1}{R} \int_0^{R/c} d\tau \rho(\mathbf{r}', t - \tau) \quad (9)$$

where $R = |\mathbf{r} - \mathbf{r}'|$ (a gauge function is defined to within an arbitrary additive constant, which we omit here). For a point charge q moving with a constant velocity $\mathbf{v}$ along the x -axis, the charge density is

$$\rho(\mathbf{r}, t) = q \delta(\mathbf{r} - \mathbf{v}t) = q \delta(x - vt) \delta(y) \delta(z). \quad (10)$$

This gives

$$\begin{aligned} \int_0^{R/c} d\tau \rho(\mathbf{r}', t - \tau) &= q \delta(y') \delta(z') \int_0^{R/c} d\tau \delta[x' - v(t - \tau)] \\ &= \frac{q}{|v|} \delta(y') \delta(z') \int_0^{R/c} d\tau \delta[\tau - (t - x'/v)] \\ &= \frac{q}{|v|} \delta(y') \delta(z') \{ \Theta[R/c - (t - x'/v)] - \Theta[-(t - x'/v)] \} \\ &= \frac{q}{v} \delta(y') \delta(z') [\Theta(x' - x_0) - \Theta(x' - vt)] \end{aligned} \quad (11)$$

where $\Theta(x)$ is the Heaviside step function and

$$x_0 = x - \gamma^2 \left[x - vt + \frac{v}{c} \sqrt{(x - vt)^2 + \frac{1}{\gamma^2}(y^2 + z^2)} \right] \quad \gamma = \frac{1}{\sqrt{1 - v^2/c^2}}. \quad (12)$$

The gauge function (9) with the charge density (10) is thus

$$\begin{aligned}
\chi_C(\mathbf{r}, t) &= -q \frac{c}{v} \int_{-\infty}^{\infty} dx' \frac{\Theta(x' - x_0) - \Theta(x' - vt)}{\sqrt{(x - x')^2 + y^2 + z^2}} \\
&= -q \frac{c}{v} \int_{x_0}^{vt} \frac{dx'}{\sqrt{(x - x')^2 + y^2 + z^2}} \\
&= q \frac{c}{v} \left[\operatorname{arcsinh} \frac{x - vt}{\sqrt{y^2 + z^2}} - \operatorname{arcsinh} \frac{x - x_0}{\sqrt{y^2 + z^2}} \right].
\end{aligned} \tag{13}$$

Let us first check that the gauge function (13) yields the established difference $V_C - V_L$ between the scalar potentials in the two gauges. Using the identity

$$\frac{1}{\gamma^2} \sqrt{(x - x_0)^2 + y^2 + z^2} = \frac{v}{c} (x - vt) + \sqrt{(x - vt)^2 + \frac{1}{\gamma^2} (y^2 + z^2)} \tag{14}$$

which follows from (12), to simplify the result of the partial differentiation $-\partial\chi_C/c\partial t$, we obtain

$$V_C - V_L = -\frac{\partial\chi_C}{c\partial t} = \frac{q}{\sqrt{(x - vt)^2 + y^2 + z^2}} - \frac{q}{\sqrt{(x - vt)^2 + \frac{1}{\gamma^2} (y^2 + z^2)}}. \tag{15}$$

This is indeed the correct result [see equations (1) and (2)].

Calculating the x -component of the difference $\mathbf{A}_C - \mathbf{A}_L$ between the vector potentials in the two gauges is now very simple because $\partial\chi_C/\partial x = -(1/v)\partial\chi_C/\partial t$ on account of the dependence of the gauge function (13) on the variables x and t only through the combination $x - vt$. We thus have

$$A_{Cx} - A_{Lx} = \frac{\partial\chi_C}{\partial x} = -\frac{c}{v} \frac{\partial\chi_C}{c\partial t} = \frac{c}{v} (V_C - V_L). \tag{16}$$

The y - and z -components of the difference $\mathbf{A}_C - \mathbf{A}_L = \nabla\chi_C$ are obtained by performing direct differentiations in a similar way to that of calculating the value (15) for the difference $V_C - V_L$, yielding

$$A_{Cy} - A_{Ly} = \frac{\partial\chi_C}{\partial y} = -\frac{c}{v} \frac{y(x - vt)}{y^2 + z^2} (V_C - V_L) \tag{17}$$

$$A_{Cz} - A_{Lz} = \frac{\partial\chi_C}{\partial z} = -\frac{c}{v} \frac{z(x - vt)}{y^2 + z^2} (V_C - V_L). \tag{18}$$

These components have no singularities (they vanish at $x - vt = y = z = 0$). As $A_{Ly} = A_{Lz} = 0$, equations (17) and (18) also give the Coulomb-gauge components A_{Cy} and A_{Cz} themselves, respectively.

It is instructive to perform the Lorentz transformation of the Coulomb-gauge four-potential $(V_C, \mathbf{A}_C)$ from the ‘laboratory’ frame, where the charge moves with the constant velocity $\mathbf{v} = v\hat{\mathbf{x}}$, to its rest frame (the primes denote the rest-frame quantities):

$$V'_C = \gamma(V_C - vA_{Cx}/c) \tag{19}$$

$$A'_{Cx} = \gamma(A_{Cx} - vV_C/c) \quad A'_{Cy} = A_{Cy} \quad A'_{Cz} = A_{Cz}. \tag{20}$$

We note first that with the x -component A_{Cx} of the vector potential in the Coulomb gauge given by

$$A_{Cx} = \frac{c}{v}(V_C - V_L/\gamma^2) \quad (21)$$

which follows from the expression (16) for the difference $A_{Cx} - A_{Lx}$ and the fact that $A_{Lx} = (v/c)V_L$, equation (19) gives the rest-frame scalar potential V'_C as

$$V'_C = \frac{1}{\gamma}V_L = \frac{q}{\gamma\sqrt{(x-vt)^2 + (y^2 + z^2)/\gamma^2}} = \frac{q}{\sqrt{x'^2 + y'^2 + z'^2}} \quad (22)$$

where the Lorentz transformation of the coordinates, $x' = \gamma(x - vt)$, $y' = y$, $z' = z$, is performed on the right-hand side. The rest-frame scalar potential V'_C is simply that of a point charge q in electrostatics. We note also that because the Coulomb-gauge condition $\nabla \cdot \mathbf{A} = 0$ is not Lorentz invariant, the rest-frame vector potential $\mathbf{A}'_C$ is not divergenceless in the rest-frame variables x' , y' , z' ; the potentials V'_C , $\mathbf{A}'_C$ are therefore no longer those of the Coulomb gauge. However, a direct calculation shows that the vector potential $\mathbf{A}'_C$ is irrotational, $\nabla' \times \mathbf{A}'_C = 0$, which expresses the fact that there is no magnetic field in the rest frame. Moreover, the vector potential $\mathbf{A}'_C$ is independent of the rest-frame time t' , and thus the electric field in the rest frame is given only by $\mathbf{E}' = -\nabla' V'_C$, which yields correctly the electrostatic Coulomb field of a charge at rest.

3. Inhomogeneous wave equation for the vector-potential difference

The difference $\mathbf{A}_C - \mathbf{A}_L$ must satisfy the inhomogeneous wave equation

$$\square(\mathbf{A}_C - \mathbf{A}_L) = \nabla \frac{\partial V_C}{c \partial t} \quad (23)$$

which is obtained by subtracting the wave equation for $\mathbf{A}_L$,

$$\square \mathbf{A}_L = -\frac{4\pi}{c} q \mathbf{v} \delta(\mathbf{r} - \mathbf{v}t) \quad (24)$$

(see [1], section 6.3) from the wave equation (3) for $\mathbf{A}_C$. It is straightforward to show that the x -component $A_{Cx} - A_{Lx} = (c/v)(V_C - V_L)$ of the difference $\mathbf{A}_C - \mathbf{A}_L$ indeed satisfies the x -component of the inhomogeneous wave equation (23):

$$\square \left[\frac{c}{v}(V_C - V_L) \right] = \frac{\partial^2 V_C}{c \partial t \partial x}. \quad (25)$$

The fact that equation (25) holds true follows directly from the wave equations

$$\frac{c}{v} \square V_C = \frac{\partial^2 V_C}{c \partial t \partial x} - \frac{4\pi c}{v} q \delta(x - vt) \delta(y) \delta(z) \quad (26)$$

and

$$\square V_L = -4\pi q \delta(x - vt) \delta(y) \delta(z). \quad (27)$$

The wave equation (26) in turn holds true because of the facts that the d'Alembertian $\square = \nabla^2 - \partial^2/c^2 \partial t^2$, $\nabla^2 V_C = -4\pi q \delta(x-vt)\delta(y)\delta(z)$, and $\partial V_C/\partial t = -v \partial V_C/\partial x$; the wave equation (27) embodies the fact that the Lorenz-gauge scalar potential V_L is the retarded solution of the inhomogeneous wave equation with the right-hand side $-4\pi q \delta(x-vt)\delta(y)\delta(z)$.

It is also straightforward to show that the y -component (17) and z -component (18) of the difference $\mathbf{A}_C - \mathbf{A}_L$ satisfy the inhomogeneous wave equation (23) by using the identity

$$\square(fg) = g\square f + f\square g + 2\nabla f \cdot \nabla g - 2 \frac{\partial f}{c \partial t} \frac{\partial g}{c \partial t} \quad (28)$$

and equation (25) in the evaluation of the requisite derivatives; the singularities of the functions $y(x-vt)/(y^2+z^2)$ and $z(x-vt)/(y^2+z^2)$ at $y = z = 0$ cannot introduce any delta-function terms in the d'Alembertians of the components (17) and (18) as the latter are functions with no singularities.

4. Poisson's equation for the vector-potential difference

Onoochin [9] attempts to solve the x -component of the inhomogeneous wave equation (23) for the difference $\mathbf{A}_C - \mathbf{A}_L$ directly by reducing it to a Poisson equation, which is a method based on the fact that the space and time partial derivatives are not independent when the source term is moving uniformly (see [3], section 19-3). In the present case, the dependence on the variables x and t is only through the combination $x - vt$, and thus the x -component of the difference $\mathbf{A}_C - \mathbf{A}_L$ can be written as $A_{Cx} - A_{Lx} = f(x - vt, y, z)$, where the function $f(x - vt, y, z)$ satisfies an inhomogeneous wave equation

$$\begin{aligned} \square f(x - vt, y, z) &= \frac{\partial^2 V_C}{c \partial t \partial x} \\ &= -q \frac{v}{c} \frac{2(x - vt)^2 - y^2 - z^2}{[(x - vt)^2 + y^2 + z^2]^{5/2}} + q \frac{4\pi v}{3c} \delta(x - vt) \delta(y) \delta(z) \end{aligned} \quad (29)$$

that can be cast as a Poisson equation on the substitutions $\partial^2/\partial x^2 - \partial^2/c^2 \partial t^2 = \partial^2/\gamma^2 \partial x^2$ and $x - vt = \chi/\gamma$, where $\gamma = (1 - v^2/c^2)^{-1/2}$:

$$\left(\frac{\partial^2}{\partial \chi^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) f(\chi/\gamma, y, z) = -q \frac{v}{c} \frac{2\chi^2/\gamma^2 - y^2 - z^2}{(\chi^2/\gamma^2 + y^2 + z^2)^{5/2}} + q \frac{4\pi v}{3c} \gamma \delta(\chi) \delta(y) \delta(z). \quad (30)$$

The delta-function term on the right-hand side of equation (29) arises from the fact that $\partial^2 V_C / c \partial t \partial x = -(v/c) \partial^2 V_C / \partial x^2$ and the delta-function identity [11]

$$\frac{\partial^2}{\partial x_i \partial x_j} \frac{1}{r} = \frac{3x_i x_j - r^2 \delta_{ij}}{r^5} - \frac{4\pi}{3} \delta_{ij} \delta(\mathbf{r}) \quad (31)$$

where $r = (x_1^2 + x_2^2 + x_3^2)^{1/2}$, δ_{ij} is the Kronecker delta symbol and $\delta(\mathbf{r}) = \delta(x_1)\delta(x_2)\delta(x_3)$ is the three-dimensional delta function. Strictly speaking, the first term on the right-hand side of equation (31) should be understood as $\lim_{a \rightarrow 0} (3x_i x_j - r^2 \delta_{ij}) / (r^2 + a^2)^{5/2}$, where the limit is to be performed after an $\mathbf{R}^3$ integration with a well-behaved 'test' function. This limit is automatically implemented when the integration is done in spherical coordinates

and the integration over the angular variables is performed first (without any additional transformation of variables that would shift the origin, of course) [11,12].

Unlike the well-known delta-function identity $\nabla^2(1/r) = -4\pi\delta(\mathbf{r})$, the identity (31) is needed only relatively rarely in electromagnetism, an example being the calculation of the fields of electric and magnetic dipoles from their potentials (see [11]; and [1], equations (4.20) and (5.64)). The identity $\nabla^2(1/r) = -4\pi\delta(\mathbf{r})$ obviously requires that the second-order partial derivatives $\partial^2 r^{-1}/\partial x_i^2$, $i = 1, 2, 3$ are given by expressions like those of equation (31); the precise form of these can be seen most easily to follow from the limit $a \rightarrow 0$ of

$$\frac{\partial^2}{\partial x_i \partial x_j} \frac{1}{\sqrt{r^2 + a^2}} = \frac{3x_i x_j - r^2 \delta_{ij}}{(r^2 + a^2)^{5/2}} - \frac{a^2 \delta_{ij}}{(r^2 + a^2)^{5/2}} \quad (32)$$

as here the limit $a \rightarrow 0$ of the second term on the right-hand side is a representation of $-\frac{4}{3}\pi\delta_{ij}\delta(\mathbf{r})$.

The standard integral expression for the solution to Poisson's equation (30) is given by

$$f = \frac{qv}{4\pi c} \int_{-\infty}^{\infty} d\chi' \int_{-\infty}^{\infty} dy' \int_{-\infty}^{\infty} dz' \left[\frac{2\chi'^2/\gamma^2 - y'^2 - z'^2}{(\chi'^2/\gamma^2 + y'^2 + z'^2)^{5/2}} - \frac{4\pi}{3} \gamma \delta(\chi') \delta(y') \delta(z') \right] \times \frac{1}{\sqrt{(\chi - \chi')^2 + (y - y')^2 + (z - z')^2}} \quad (33)$$

where one recovers the original variables by putting $\chi = \gamma(x - vt)$. Here, the delta-function term is readily integrated to yield

$$f = \frac{qv}{4\pi c} \int_{-\infty}^{\infty} d\chi' \int_{-\infty}^{\infty} dy' \int_{-\infty}^{\infty} dz' \frac{2\chi'^2/\gamma^2 - y'^2 - z'^2}{(\chi'^2/\gamma^2 + y'^2 + z'^2)^{5/2}} \frac{1}{\sqrt{(\chi - \chi')^2 + (y - y')^2 + (z - z')^2}} - \frac{qv\gamma}{3c} \frac{1}{\sqrt{\chi^2 + y^2 + z^2}}. \quad (34)$$

A direct evaluation in closed form of the three-dimensional integral in (34) does not seem possible. However, this integral can be evaluated for the special case $y = z = 0$, and the result is (see Appendix):

$$\begin{aligned} \frac{qv}{4\pi c} \int_{-\infty}^{\infty} d\chi' \int_{-\infty}^{\infty} dy' \int_{-\infty}^{\infty} dz' \frac{2\chi'^2/\gamma^2 - y'^2 - z'^2}{(\chi'^2/\gamma^2 + y'^2 + z'^2)^{5/2}} \frac{1}{\sqrt{(\chi - \chi')^2 + y'^2 + z'^2}} \\ = \frac{qv\gamma}{3c|\chi|} = \frac{qv}{3c|x - vt|}. \end{aligned} \quad (35)$$

For $y = z = 0$, the value of the second term on the right-hand side of (34) is exactly equal and opposite to the value of (35), and thus the solution (34) of the Poisson equation (30) vanishes at $y = z = 0$, as required by the result (16) that we obtained for the difference $A_{Cx} - A_{Lx}$.

Onoochin's calculation omits the delta-function term that arises from the second-order partial derivative of the Coulomb-gauge scalar potential V_C , and he takes erroneously just the first term of (34) as the solution of the Poisson equation (30). According to equation (35), this incorrect solution is non-zero at $y = z = 0$, and, moreover, its partial time derivative

does not vanish there. As $\partial(V_C - V_L)/\partial x = 0$ at $y = z = 0$, the result (35), if it were the true difference $A_{Cx} - A_{Lx}$ at $y = z = 0$, would lead there to a non-zero difference between the Coulomb- and Lorenz-gauge x -components of the electric field. This is Onoochin's evidence against the equivalence of the Lorenz and Coulomb gauges. (In his paper [9], Onoochin does not evaluate the integral in equation (34), but he has communicated to the present author a calculation of its value at $y=z=0$ that differs from the value given by equation (35) by a factor of 3 [13].)

It is not difficult to show directly that

$$f = \frac{c}{v} \left(\frac{q}{\sqrt{\chi^2/\gamma^2 + y^2 + z^2}} - \frac{q\gamma}{\sqrt{\chi^2 + y^2 + z^2}} \right) \quad (36)$$

is the vanishing-at-infinity solution of the Poisson equation (30). Using the relation (31) and the differentiation rule $\partial^2 g(ax, y, z)/\partial x^2 = a^2 \partial^2 g(u, y, z)/\partial u^2|_{u=ax}$, we have

$$\frac{\partial^2}{\partial \chi^2} \frac{1}{\sqrt{\chi^2/\gamma^2 + y^2 + z^2}} = \frac{1}{\gamma^2} \left[\frac{2\chi^2/\gamma^2 - y^2 - z^2}{(\chi^2/\gamma^2 + y^2 + z^2)^{5/2}} - \frac{4\pi}{3} \gamma \delta(\chi) \delta(y) \delta(z) \right] \quad (37)$$

$$\frac{\partial^2}{\partial y^2} \frac{1}{\sqrt{\chi^2/\gamma^2 + y^2 + z^2}} = \frac{2y^2 - \chi^2/\gamma^2 - z^2}{(\chi^2/\gamma^2 + y^2 + z^2)^{5/2}} - \frac{4\pi}{3} \gamma \delta(\chi) \delta(y) \delta(z) \quad (38)$$

$$\frac{\partial^2}{\partial z^2} \frac{1}{\sqrt{\chi^2/\gamma^2 + y^2 + z^2}} = \frac{2z^2 - \chi^2/\gamma^2 - y^2}{(\chi^2/\gamma^2 + y^2 + z^2)^{5/2}} - \frac{4\pi}{3} \gamma \delta(\chi) \delta(y) \delta(z) \quad (39)$$

which gives

$$\begin{aligned} & \left(\frac{\partial^2}{\partial \chi^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \frac{1}{\sqrt{\chi^2/\gamma^2 + y^2 + z^2}} \\ &= - \left(1 - \frac{1}{\gamma^2} \right) \frac{2\chi^2/\gamma^2 - y^2 - z^2}{(\chi^2/\gamma^2 + y^2 + z^2)^{5/2}} - \frac{4\pi}{3} \left(2 + \frac{1}{\gamma^2} \right) \gamma \delta(\chi) \delta(y) \delta(z). \end{aligned} \quad (40)$$

Using the relation $\nabla^2(1/r) = -4\pi\delta(\mathbf{r})$, we have also

$$\left(\frac{\partial^2}{\partial \chi^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \frac{\gamma}{\sqrt{\chi^2 + y^2 + z^2}} = -4\pi\gamma\delta(\chi)\delta(y)\delta(z). \quad (41)$$

Subtracting equation (41) from equation (40), multiplying the result by qc/v and using the fact that $1 - \gamma^{-2} = v^2/c^2$, we obtain

$$\begin{aligned} & \left(\frac{\partial^2}{\partial \chi^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \frac{c}{v} \left(\frac{q}{\sqrt{\chi^2/\gamma^2 + y^2 + z^2}} - \frac{q\gamma}{\sqrt{\chi^2 + y^2 + z^2}} \right) \\ &= -q \frac{v}{c} \frac{2\chi^2/\gamma^2 - y^2 - z^2}{(\chi^2/\gamma^2 + y^2 + z^2)^{5/2}} + q \frac{4\pi v}{3c} \gamma \delta(\chi) \delta(y) \delta(z) \end{aligned} \quad (42)$$

which shows that the function f of equation (36) indeed satisfies the Poisson equation (30). Transforming back to the original variables through $\chi = \gamma(x - vt)$, the function f becomes

$$f = \frac{c}{v} \left[\frac{q}{\sqrt{(x-vt)^2 + y^2 + z^2}} - \frac{q}{\sqrt{(x-vt)^2 + \frac{1}{\gamma^2}(y^2 + z^2)}} \right] = \frac{c}{v}(V_C - V_L) \quad (43)$$

which is the value (16) for the difference $A_{Cx} - A_{Lx}$.

The function f of equation (36) has to equal the integral solution (34), yielding for the integral in (34) a closed-form expression

$$\begin{aligned} \frac{1}{4\pi} \int_{-\infty}^{\infty} d\chi' \int_{-\infty}^{\infty} dy' \int_{-\infty}^{\infty} dz' \frac{2\chi'^2/\gamma^2 - y'^2 - z'^2}{(\chi'^2/\gamma^2 + y'^2 + z'^2)^{5/2}} \frac{1}{\sqrt{(\chi-\chi')^2 + (y-y')^2 + (z-z')^2}} \\ = \frac{c^2}{v^2} \left[\frac{1}{\sqrt{\chi^2/\gamma^2 + y^2 + z^2}} - \left(1 - \frac{v^2}{3c^2}\right) \frac{\gamma}{\sqrt{\chi^2 + y^2 + z^2}} \right]. \end{aligned} \quad (44)$$

Direct numerical three-dimensional quadrature of this integral, with the first factor of the integrand regularized as mentioned above in connection with equation (31), resulted in values that were close to those of the closed-form expression on the right-hand side of (44).

5. Concluding remarks

We found an explicit expression for the gauge function of the transformation between the Lorenz and Coulomb gauges for a uniformly moving point charge. The Coulomb-gauge potentials obtained using the gauge function are guaranteed to yield the same electric and magnetic fields as the well-known Lorenz-gauge potentials of the charge. The expression obtained for the difference between the vector potentials in the two gauges satisfies the Poisson equation to which the inhomogeneous wave equation for this difference reduces after a transformation of the variables. However, the right-hand side of the Poisson equation involves a delta-function term which has to be included in the integral expression for its solution.

Although gauge invariance is a foregone conclusion in a gauge-invariant theory, we believe that an explicit demonstration of the equivalence of the Lorenz and Coulomb gauges in the basic problem of a uniformly moving charge was instructive.

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Appendix

After the transformation $\chi'/\gamma = x'$, the integral (35) can be written as $(qv\gamma/c)I(\gamma, X)$, where $I(\gamma, X)$ is the integral (we drop the primes on the integration variables):

$$I(\gamma, X) = \frac{1}{4\pi} \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dz \frac{2x^2 - y^2 - z^2}{(x^2 + y^2 + z^2)^{5/2}} \frac{1}{\sqrt{\gamma^2(X-x)^2 + y^2 + z^2}} \quad (45)$$

where $X = \chi/\gamma$ and $\gamma > 1$ are real parameters.

Transforming from the Cartesian coordinates x, y, z to the spherical ones, r, θ, ϕ , we have

$$\begin{aligned}
I(\gamma, X) &= \frac{1}{2} \int_0^\infty r^2 dr \int_0^\pi \sin \theta d\theta \frac{r^2(3 \cos^2 \theta - 1)}{r^5} \frac{1}{\sqrt{\gamma^2(X - r \cos \theta)^2 + r^2(1 - \cos^2 \theta)}} \\
&= \frac{1}{2} \int_0^\infty dr \int_{-1}^1 d\xi \frac{3\xi^2 - 1}{r} \frac{1}{\sqrt{\gamma^2(X - r\xi)^2 + r^2(1 - \xi^2)}}.
\end{aligned} \tag{46}$$

We integrate with respect to ξ first (this is needed to implement the limit $a \rightarrow 0$ of the first term on the right-hand side of equation (32), see [11,12]):

$$\begin{aligned}
F(r, \gamma, X) &= \int_{-1}^1 d\xi \frac{3\xi^2 - 1}{r} \frac{1}{\sqrt{\gamma^2(X - r\xi)^2 + r^2(1 - \xi^2)}} \\
&= \frac{1}{2\omega^5 r^4} \left[3\gamma\omega(A+B) + (1+2\gamma^2)(\omega^2 r^2 - 3\gamma^2 X^2) \ln \frac{\gamma\omega|X+r| - \omega^2 r - \gamma^2 X}{\gamma\omega|X-r| + \omega^2 r - \gamma^2 X} \right]
\end{aligned} \tag{47}$$

where

$$\omega = \sqrt{\gamma^2 - 1} \quad A = (\omega^2 r - 3\gamma^2 X)|X + r| \quad B = (\omega^2 r + 3\gamma^2 X)|X - r|. \tag{48}$$

The function $F(r, \gamma, X)$ has the properties

$$F(r, \gamma, X) = F(r, \gamma, -X) \quad \lim_{r \rightarrow 0^+} F(r, \gamma, X) = 0. \tag{49}$$

It peaks at $r = |X|$, where its derivative with respect to r is discontinuous. The argument of the logarithm in (47) reduces to $(\gamma|X| - \omega r)/(\gamma|X| + \omega r)$ for $r < |X|$, and to $(\gamma - \omega)/(\gamma + \omega) = (\gamma - \omega)^2$ for $r > |X|$.

The integration with respect to r is performed in two parts:

$$I_1(\gamma, X) = \frac{1}{2} \int_0^{|X|} dr F(r, \gamma, X) = \frac{(2 + \gamma^2)(1 + 2\gamma^2)}{6\gamma\omega^4|X|} + \frac{1 + 2\gamma^2}{2\omega^5|X|} \ln(\gamma - \omega) \tag{50}$$

$$I_2(\gamma, X) = \frac{1}{2} \int_{|X|}^\infty dr F(r, \gamma, X) = -\frac{3\gamma}{2\omega^4|X|} - \frac{1 + 2\gamma^2}{2\omega^5|X|} \ln(\gamma - \omega) \tag{51}$$

where again $\omega = (\gamma^2 - 1)^{1/2}$. The whole integral $I(\gamma, X)$ is thus

$$I(\gamma, X) = I_1(\gamma, X) + I_2(\gamma, X) = \frac{1}{3\gamma|X|} \tag{52}$$

and the integral (35) has the value $(qv\gamma/c)I(\gamma, X) = qv/(3c|X|) = qv\gamma/(3c|\chi|)$.

The results of this Appendix were obtained using the software system *Mathematica* [14], and were checked by performing numerical integrations.

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