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MATHEMATICAL MODEL OF A HEAD SUBJECTED TO AN ANGULAR ACCELERATION*

G. N. BYCROFT
Stanford Research Institute, Menlo Park, California 94025, U.S.A.†

Abstract—The brain is modelled as a spherical viscoelastic medium attached to the inside of a rigid spherical shell. The mechanical wave motion which is generated in the brain when the head is subjected to an angular acceleration of a particular amplitude and duration is calculated and the results are compared with experimental data on monkeys. The experimental data show that the threshold of angular acceleration necessary to cause concussion varies with the duration of the acceleration in a manner predicted approximately by the model. The "two-thirds power scaling law" is directly established for an elastic model but is shown to be deficient for a viscoelastic model of the brain. A value of shear strain of 0·05 in the region of the upper reticular formation is deduced as that necessary to cause concussion. The model predicts that an angular acceleration greater than $3{\cdot}5 \times 10^3$ rad/sec^2 is necessary to produce concussion in man.

INTRODUCTION

HEAD INJURIES range from superficial scalp lacerations to fatal rupture of skull and brain. 'Concussion,' which is loosely defined in this paper as loss of consciousness, may be regarded as a lower bound to serious head injury. Loss of consciousness may be caused by many factors. Among these are mechanical motions imparted to the skull and hence to the brain causing impairment of brain operation. The mechanical properties of brain tissue are those of a viscous solid having a high compression modulus close to that of water and having a very small shear modulus. Consequently it appears probable that certain motions of the head may generate large shear strains in the brain and these shear strains may disturb the brain function.

This theory of brain damage was first suggested by Holbourn (1943). A blow to the head causes both translation and rotation of the head. Mechanical waves propagate from the site of the blow through the skull and the brain matter and reflect internally leading to the formation of tensile and shear waves and possible rupture of the brain material. The rotational component of the head motion generates shear waves directly. Rotation of the head may also be generated indirectly by 'whiplash' accidents.

There is much controversy as to the relative importance of the translational and rotational components of the head motion. However, it is known Ommaya *et al.* (1968) and Unterharnscheidt *et al.* (1969) that a nondeforming angular acceleration of the head will cause concussion. These investigators demonstrated concussion resulting from angular accelerations by experiments on monkeys. Ommaya performed whiplash experiments on rhesus monkeys and Unterharnscheidt used a mechanism to give a prescribed angular acceleration to the head of squirrel monkeys.

Consequently the following mathematical model of a head subjected to an angular acceleration has been constructed in order to examine the shear strains developed in the brain.

MATHEMATICAL MODEL

The shear modulus of the skull is very large compared to the shear modulus of the brain

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†Now at Physics and Engineering Lab., D.S.I.R., Lower Hutt, New Zealand.

and for the case of axisymmetric rotation the skull is treated as a rigid spherical shell. The brain material is treated as a linearly visco-elastic medium having a shear modulus very small compared to the compression modulus.

The inside of an animal skull is irregular and it is assumed that an angular acceleration of the skull will be imparted directly to the outer surface of the brain; that is, it is assumed that the outer surface of the brain moves in shear with the inner surface of the skull. This motion of the outer surface of the brain will generate mechanical shear waves which will propagate to the center of the brain and reflect back and forth. Initially the elastic case is solved and then the viscoelastic solution is deduced from the elastic solution by the 'correspondence principle.'

Elastic case

ρ = density of brain matter, lb/ft^3
μ = shear modulus of brain matter, pdls/ft^2
r = radial distance of a point from brain center, ft
w = shear displacement of a point (r, θ), ft
θ = angular position of a point at radius r, rad
a = radius of brain, ft.

It may be readily shown, Love (1927) that the equation of motion of this system is given by

$$\frac{\rho\partial^2 w}{\mu\partial t^2} = \frac{1}{r}\left[r\frac{\partial^2 w}{\partial r^2} + \frac{2\partial w}{\partial r} + \frac{1}{r}\frac{\partial^2 w}{\partial\theta^2} - \frac{w}{r}\frac{1}{\sin^2\theta} + \frac{\cot\theta}{r}\frac{\partial w}{\partial\theta}\right]. \quad (1)$$

This equation is solved for the particular boundary conditions of the problem in the following manner. The Laplace transform of equation (1) is given by,

$$\frac{\rho}{\mu}s^2\bar{w} = \frac{1}{r}\left[r\frac{\partial^2\bar{w}}{\partial r^2} + 2\frac{\partial\bar{w}}{\partial r} + \frac{1}{r}\frac{\partial^2\bar{w}}{\partial\theta^2} - \frac{\bar{w}}{r}\frac{1}{\sin^2\theta} + \frac{\cot\theta}{r}\frac{\partial\bar{w}}{\partial\theta}\right], \quad (2)$$

where $\bar{w}$ is the Laplace transform of w and s is the transform variable. A separation of variables results in the solution to equation (2) in the form

$$\bar{w} = \frac{A}{iscr}\left[\frac{\sin(iscr)}{iscr} - \cos(iscr)\right]\sin\theta, \quad (3)$$

where A is an arbitrary function and

$$c = \sqrt{\frac{\rho}{\mu}}, \quad (4)$$

that is, the reciprocal of the shear wave velocity.

Consider the skull and hence the outer brain surface $r = a$ being given an angular acceleration consisting of a single sine wave pulse described by

$$\begin{aligned} \ddot{\Phi} &= B\sin\Omega t & 0 < t < \frac{\Pi}{\Omega} \\ &= 0 & \frac{\Pi}{\Omega} < t. \end{aligned} \quad (5)$$

The Laplace transform of the surface acceleration of the brain follows readily as

$$\bar{\bar{w}} = \frac{aB\Omega}{(s^2+\Omega^2)}\left[1 + e^{-(\Pi/\Omega)s}\right], \quad (6)$$

which gives

$$A = \frac{ica^2\Omega\left[1 + e^{-(\Pi/\Omega)s}\right]B}{s\left[\frac{\sinh(sca)}{sca} - \cosh(sca)\right](s^2+\Omega^2)} \quad (7)$$

and

$$\bar{w} = \frac{Ba^2\Omega\left[1 + e^{-\pi s/\Omega}\right]\left[\frac{\sinh(scr)}{scr} - \cosh(scr)\right]\sin\theta}{s^2\left[\frac{\sinh(sca)}{sca} - \cosh(sca)\right](s^2+\Omega^2)r}. \quad (8)$$

We are particularly interested in the shear strain. The transform of the shear strain is given by

$$\bar{\epsilon} = \mu\left[\frac{\partial \bar{w}}{\partial r} - \frac{\bar{w}}{r}\right]. \qquad (9)$$

From equations (8 and 9)

$$\bar{\epsilon} = \frac{Ba^3\Omega}{r^3}\,\frac{(1+e^{-(\Pi/\Omega s)})[3scr\cosh(scr) - 3\sinh(scr) - s^2c^2r^2\sinh(scr)]}{s^2(s^2+\Omega^2)[\sinh(sca) - sca\cosh(sca)]}. \qquad (10)$$

If this equation is inverted and non-dimensionalized by putting

$$r/a = \lambda,\ t/ca = \tau,\ s = x/ca,\ \Omega_o = \Omega ca, \qquad (11)$$

then

$$\epsilon = \frac{Ba^2\Omega_o c^2}{\lambda^3}\int_{c-i\infty}^{c+i\infty}\frac{(1+e^{-\Pi x/\Omega_o})[3\lambda x\cosh(\lambda x) - 3\sinh(\lambda x) - \lambda^2x^2\sinh(\lambda x)]e^{x\tau}\,dx}{x^2(x^2+\Omega_o^2)(\sinh x - x\cosh x)}. \qquad (12)$$

Scaling law. Equation (12) shows that two different-sized brains of the same material and the same values of λ and Ω_o experience shear strains proportional to the square of the brain radius. The square of the brain radius is proportional to the 2/3 power of the brain mass and consequently equation (12) establishes the scaling law proposed by Holbourn (1943). This scaling law may be stated more specifically for an elastic brain as: the same shear stress or strain will be developed in two different sized brains of the same material at radii proportional to the brain dimensions when subjected to impulses of the same shape but of durations inversely proportional to the brain dimensions and amplitudes inversely proportional to the 2/3 power of the brain mass.

Viscoelastic case

The viscoelastic case is derived from the elastic case by use of the 'correspondence principle.' This principle states that the transform of the viscoelastic solution is obtained by replacing the elastic compliance by the viscoelastic complex compliance in the transform of the elastic solution.

The general linear viscoelastic complex compliance $J(s)$ in terms of the transform variable s is given by

$$J(s) = \frac{p_ns^n + p_{n-1}s^{n-1} + \qquad + p_o}{q_{n+1}s^{n+1} + q_ns^n + \qquad + q_o}. \qquad (13)$$

This expression includes the various Maxwell, Kelvin, and higher order models of a linear viscoelastic material. In practice one endeavors to determine values of the parameters p_n and q_n to adequately fit the complex compliance to experimental data over the relevant frequency range.

If the complex compliance given by equation (13) is inserted in place of the elastic compliance in equation (10) and equation (10) is inverted and nondimensionalized. the strain in the general viscoelastic case is obtained as

$$\epsilon = \frac{Ba^2\Omega_o c_o^2}{2\pi i\lambda^3}\int_{c-i\infty}^{c+i\infty}\frac{(1+e^{-\pi x/\Omega_o})\left[\begin{array}{l}3\lambda x\sqrt{J_o(x)}\cosh\lambda x\sqrt{J_o(x)} - 3\sinh\lambda x\sqrt{J_o(x)}\\ -\lambda^2x^2J_o(x)\sinh\lambda x\sqrt{J_o(x)}\end{array}\right]e^{x\tau}\,d\tau}{x^2(x^2+\Omega_o^2)[\sinh x\sqrt{J_o(x)} - x\sqrt{J_o(x)}\cosh x\sqrt{J_o(x)}]} \qquad (14)$$

where

$$J_o(x) = \frac{\rho J(x/c_o a)}{c_o^2} \qquad (15)$$

and c_o^{-1} is a characteristic velocity.

The integral in equation (14) may be evaluated by integration around the Bromwich contour. The integral is treated in two parts corresponding to the parts of the factor $(1+e^{-\pi x/\Omega_o})$. This is necessary in order to obtain vanishing of the integral around the left side of the complex plane. The portion having the exponential coefficient is deduced from the portion having unity coefficient by convolution. The integrand is even with respect to the branch points at $\sqrt{J_o(x)} = 0$ and consequently is unaffected by sign changes. There are poles at $x = \pm i\Omega_o$ and at the roots of the equation

$$\sinh x\sqrt{J_o(x)} - x\sqrt{J_o(x)} \cosh x\sqrt{J_o(x)} = 0. \quad (16)$$

An expansion in terms of x where x is small shows that $x = 0$ is not a pole.

The roots of equation (16) may be obtained by considering the equation

$$\sinh y - y \cosh y = 0. \quad (17)$$

The roots of this equation are known and tabulated as $\pm iy_n$ where y_n are real numbers. Then the roots x_n are found as

$$x_n^2 J_o(x_n) + y_n^2 = 0. \quad (18)$$

This equation is a polynomial whose order depends on the function $J(s)$. In general, equation (18) has complex roots. These complex roots correspond to damping of the elastic waves. To evaluate equation (14) $J(s)$ must be given a specific form. Shuck *et al.* (1970) and Shuck and Advani (1972) measured values of the complex impedance of human brain tissue. In particular Shuck and Advani (1972) fitted various viscoelastic models to the experimental data. Figure 1 shows Shuck's experimental values of the J_1 and J_2 as a function of frequency together with the values J_{m1} and J_{m2} corresponding to a Kelvin model having the parameters shown. The fit given by this Kelvin model over the frequency range shown is adequate for the purposes of this paper. The parameters used by Shuck in his Kelvin model are different from those used here resulting in a better fit at the high frequency end and a worse fit at the low frequency end.

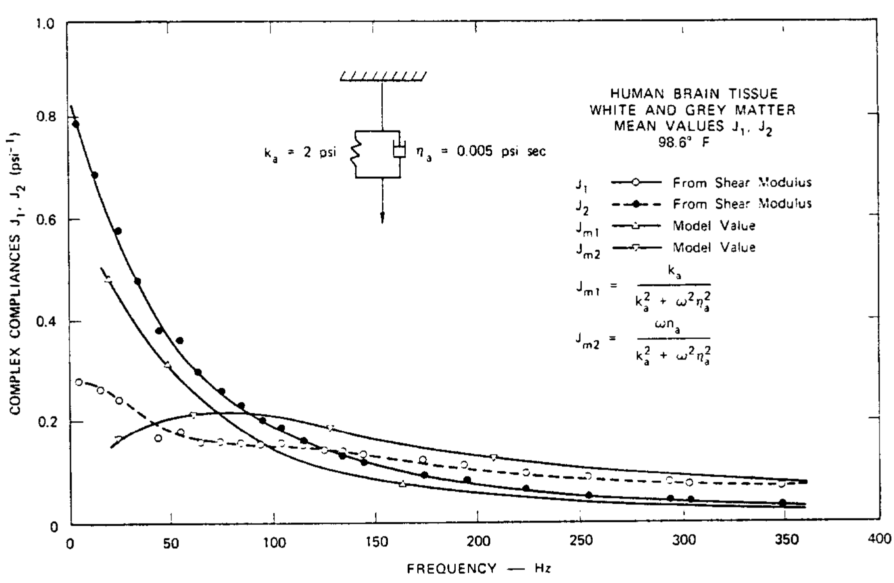

Fig. 1. Complex compliance functions of human brain tissue in shear compared with experimental values.

Equation (13) now has the simplified form given by

$$J(s) = \frac{1}{(k_a + sn_a)} \tag{19}$$

and equation (14) becomes

$$\epsilon = \frac{Ba^2\Omega_o c_o^2}{2\pi i\lambda^3} \times \int_{c-i\infty}^{c+i\infty} \frac{(1+e^{-\Pi x/\Omega_o})\begin{bmatrix}(3\lambda x/\sqrt{1+\beta_o x})\cosh(\lambda x/\sqrt{1+\beta_o x}) - 3\sinh(\lambda x/\sqrt{1+\beta_o x})\\ [-\lambda^2 x^2/(1+\beta_o x)]\sinh(\lambda x/\sqrt{1+\beta_o x})\end{bmatrix} e^{x\tau}\,dx}{x^2[x+\Omega_o^2][\sinh(x/\sqrt{1+\beta_o x}) - (x/\sqrt{1+\beta_o x})\cosh(x/\sqrt{1+\beta_o x})]} \tag{20}$$

where

$$\beta_o = \frac{\eta_a}{c_o a k_a} \tag{21}$$

and

$$c_o = \sqrt{\frac{\rho}{k_a}}. \tag{22}$$

The singularities $x_{n_{1,2}}$ are given by

$$x_{n_1} = \frac{-\beta_o y_n^2 + \sqrt{\beta_o^2 y_n^4 - 4y_n^2}}{2}$$
$$x_{n_2} = \frac{-\beta_o y_n^2 - \sqrt{\beta_o^2 y_n^4 - 4y_n^2}}{2}. \tag{23}$$

For the lower values of n these are complex numbers.

If equation (20) is written as

$$\epsilon = Ba^2 c_o^2 I, \tag{24}$$

then the contour integration discussed earlier gives I in the form

$$I = \frac{\Omega_o}{\lambda^3}\left[\frac{-F_1(i\Omega_o)E_1}{2i\Omega_o^3 F_2(i\Omega_o)} + \frac{F_1(-i\Omega_o)E_2}{2i\Omega_o^3 F_2(-i\Omega_o)} - \sum_{n=1}^{\infty} \frac{\begin{bmatrix}3\lambda y_n\cos(\lambda y_n) - 3\sin(\lambda y_n)\\ +\lambda^2 y_n^2 \sin(\lambda y_n)\end{bmatrix} \times \begin{bmatrix}\dfrac{2x_{n_1}D_1}{(x_{n_1}^2+\Omega_o^2)(2+\beta_o x_{n_1})}\\ +\dfrac{2x_{n_2}D_2}{(x_{n_2}^2+\Omega_o^2)(2+\beta_o x_{n_2})}\end{bmatrix}}{y_n^4 \sin y_n}\right] \tag{25}$$

where

$$F_1(x) = \frac{3\lambda x}{\sqrt{1+\beta_o x}}\cosh\frac{\lambda x}{\sqrt{1+\beta_o x}} - 3\sinh\frac{\lambda x}{\sqrt{1+\beta_o x}} - \frac{\lambda^2 x^2}{(1+\beta_o x)} \times \sinh\frac{\lambda x}{\sqrt{1+\beta_o x}} \tag{26}$$

$$F_2(x) = \sinh\frac{x}{\sqrt{1+\beta_o x}} - \frac{x}{\sqrt{1+\beta_o x}} \times \cosh\frac{x}{\sqrt{1+\beta_o x}} \tag{27}$$

$$E_1 = e^{i\Omega_o\tau}, \quad \tau < \frac{\Pi}{\Omega_o} \tag{28}$$

$$E_2 = e^{-i\Omega_o\tau}, \quad \tau < \frac{\Pi}{\Omega_o} \tag{29}$$

$$D_1 = e^{x_{n_1}\tau}, \quad \tau < \frac{\Pi}{\Omega_o} \tag{30}$$

$$D_2 = e^{x_{n_2}\tau}, \quad \tau < \frac{\Pi}{\Omega_o} \tag{31}$$

$$E_1 = 0, \quad \tau > \frac{\Pi}{\Omega_o} \tag{32}$$

$$E_2 = 0, \quad \tau > \frac{\Pi}{\Omega_o} \tag{33}$$

$$D_1 = e^{x_{n_1}\tau} + e^{x_{n_1}(\tau - \Pi/\Omega_o)}, \quad \tau > \frac{\Pi}{\Omega_o} \tag{34}$$

$$D_2 = e^{x_{n_2}\tau} + e^{x_{n_2}(\tau - \Pi/\Omega_o)}, \quad \tau > \frac{\Pi}{\Omega_o}. \tag{35}$$

This expression for I is an involved function of complex numbers. It is not practical to extract the real part by algebraic transformations. Consequently a computer program was written in terms of complex arithmetic and the computer was used to evaluate the expressions directly.

Figures 2, 3 and 4 show the maximum strain function I_m which is the maximum value of the function I with respect to time. The value I_m is shown as a function of Ω_0, λ, and β_0. The value β_0 given by equation (21) is a function of the brain radius and consequently is different for man, squirrel, and rhesus monkeys. Consequently the 2/3 power scaling law applies only approximately for the viscoelastic model. The same value of the viscoelastic constants shown in Fig. 1 for humans has been used for monkeys. A radius of 0·70 in. was taken for the squirrel monkey, 1·2 in. for the rhesus monkey, and 3 in. for man. In Fig. 5 the maximum strain function is shown as a function of λ. It is noticed that the

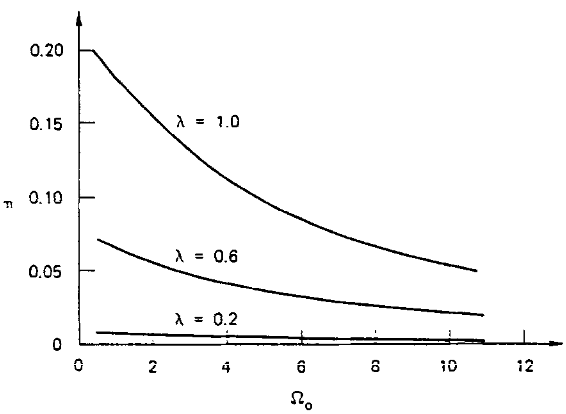

Fig. 2. $\beta_0 = 0{\cdot}525$, squirrel monkey, maximum strain function I_m.

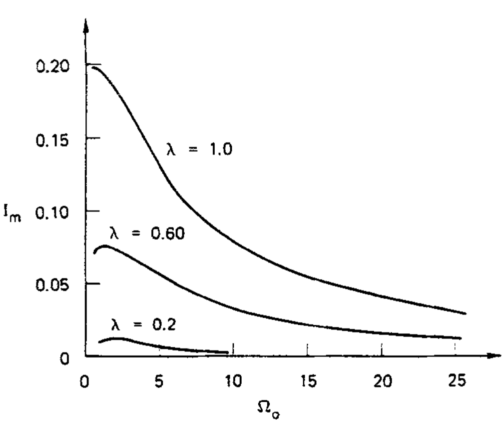

Fig. 3. $\beta_o = 0{\cdot}306$, rhesus monkey, maximum strain function I_m.

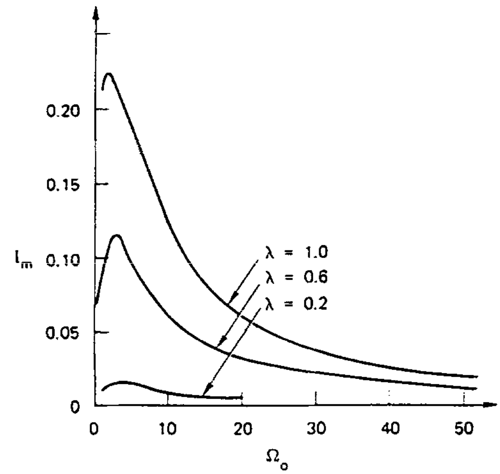

Fig. 4. $\beta_o = 0{\cdot}122$, man, maximum strain function I_m.

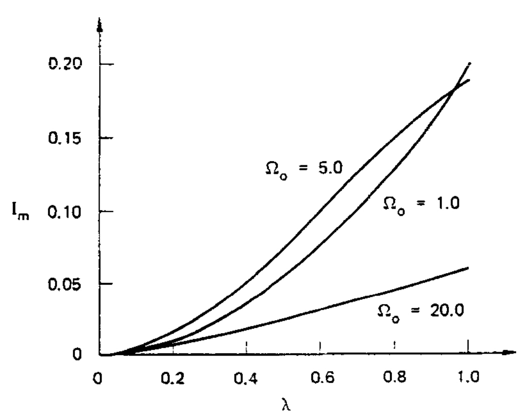

Fig. 5. $\beta_o = 0{\cdot}122$, man, maximum strain function I_m.

shear strain decreases rapidly towards the center.

COMPARISON WITH EXPERIMENTS

To compare the theory with experimental data, one must have a theory as to what causes concussion. The theory that will be used is that of Holbourn (1943) which relates brain damage to shear strain. Thomas (1970) discusses the area of the brain that controls consciousness and places it in the region of the upper reticular formation. This region is toward the midbrain and for our spherical model will be located approximately at the radius given by $\lambda = 0{\cdot}30$. Thus it is assumed that loss of consciousness will occur when a certain shear strain in the brain is exceeded at the location $\lambda = 0{\cdot}30$. The value of this shear strain may be determined by correlating the experimental work of Ommaya (1968) and Unterharnscheidt (1969) with equation (24).

In the work of Ommaya the angular acceleration is reported as the duration and amplitude of a positive half-sine wave application of acceleration. Of course, there has to be a negative acceleration applied also in order that the head comes to rest. This is not reported and may be assumed to be of longer duration and smaller amplitude than the positively applied acceleration. Unterharnscheidt applies both a positive and negative angular acceleration to the head. These two phases are stated to have the form of triangles with the negative phase of longer duration and smaller amplitude than the positive phase. However, recordings of the two phases show a rather rounded and irregular 'triangle' and in what follows these shapes are treated as half-sine waves of a specific amplitude and duration.

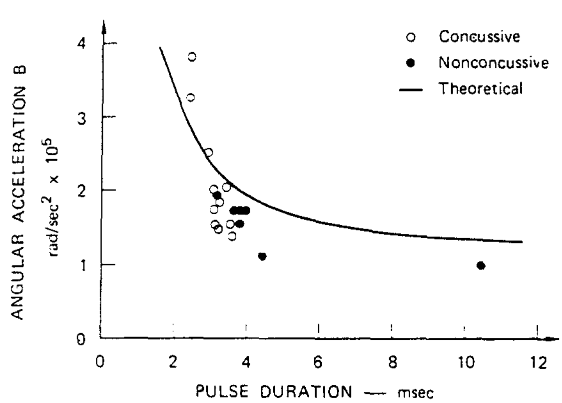

Fig. 6. Threshold of concussion; squirrel monkey, $\beta_o = 0{\cdot}525$.

Figure 6 shows the experimental data obtained by Unterharnscheidt on squirrel monkeys and Fig. 7 shows the experimental data obtained by Ommaya on rhesus monkeys.

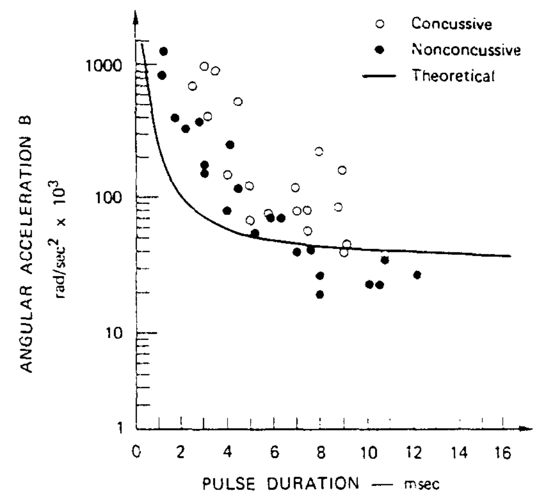

Fig. 7. Threshold of concussion; rhesus monkey, $\beta_o = 0{\cdot}306$.

The theoretical curves in Figs. 6 and 7 were obtained from equation (24) in the form,

$$B = \frac{\epsilon}{a^2 c_o^2 I_m(\lambda)} \tag{36}$$

where ϵ has been given the value of $0{\cdot}05$ at $\lambda = 0{\cdot}30$. With this value of shear strain, the theoretical curves given by equation (36) correlate reasonably well with experimental values for monkeys of considerably different size.

The correlation is best for the longer duration inputs. It is likely that the boundary conditions which assumes that the outside surface of the brain moves with the skull is not satisfied as well at short duration inputs as at long duration inputs. This may be because of the much smaller angular rotation associated with short duration inputs of a certain acceleration amplitude. The brain will not couple to the skull as well for the smaller angles of rotation because a certain amount of free motion is available.

Equation (36) was used to plot Fig. 8 showing the theoretical threshold of concussion for man assuming that lack of consciousness occurs in man when a shear strain of 0·05 at $\lambda = 0{\cdot}30$ is reached. Figure 8 was plotted out to large pulse durations and it is noticed that a minimum value of B occurs around 20 msec and that the value of this minimum angular acceleration is $3{\cdot}5\times 10^3$ rad/sec². Angular accelerations greater than $3{\cdot}5\times 10^3$ rad/sec² are thus necessary to cause concussion in man. Figure 9 shows the maximum shear strain occurring along a brain radius (man) at the threshold of concussion $\epsilon = 0{\cdot}05$ at $\lambda = 0{\cdot}30$. Relatively large shear strains are indicated at the outside of the brain but these rapidly diminish towards the center. This shows the possibility that angular rotations which are below the concussive level may still cause brain damage in the outer regions of the brain. Figure 9 also shows that in the region $0 < \lambda < 0{\cdot}50$ there is little difference in the value of the shear strain ϵ for different values of Ω_0. This means that if values of λ in the region $0 < \lambda < 0{\cdot}50$ other than $\lambda = 0{\cdot}30$ were taken as the location where consciousness is controlled, the results of Figs. 6, 7 and 8 would be little affected.

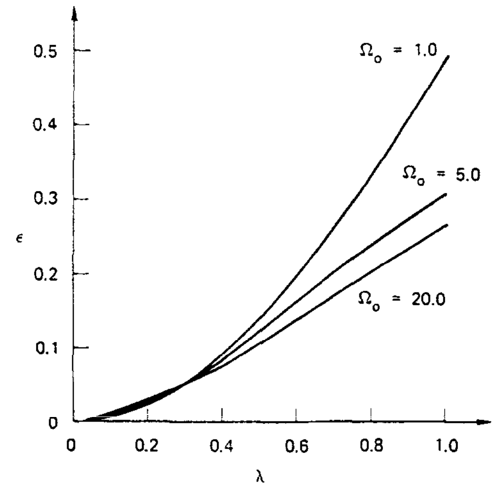

Fig. 9. Theoretical strain through brain (man) at threshold of concussion.

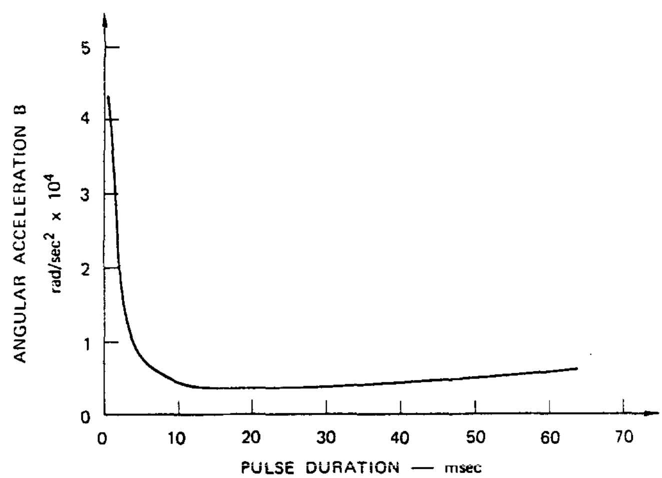

Fig. 8. Theoretical threshold of concussion; man, $\beta_0 = 0{\cdot}122$.

CONCLUDING REMARKS

This mathematical model of a head subjected to an angular rotation has behavior similar to that observed in experiments on monkeys. A value of shear strain necessary to cause concussion has been deduced from the model used in conjunction with experimental work on monkeys. If the same value of shear strain of 0·05 in the region $\lambda = 0{\cdot}30$ could be readily developed by direct blows to the head not involving head rotation, then a case could be made for the importance of translational motions of head impact.

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